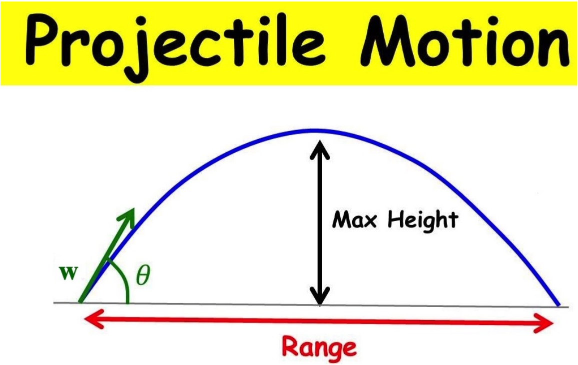
MATH 223: Multivariable Calculus



Class 4: September 19, 2022

Notes on Assignment 3
Assignment 4

Review: Curves and Tangent Lines

Example 1: $\mathbf{F}(x) = \left(x^3 + 7x + 3, 8 + \sin x\right), -2 \leq x \leq 2.$

$$\mathbf{F}(0) = (3, 8)$$

So $\mathbf{F}'(x) = (3x^2 + 7, \cos x)$

Implying $\mathbf{F}'(0) = (7, 1)$

Tangent Line

$$\begin{aligned} \mathbf{L}(t) &= \mathbf{F}(0) + t\mathbf{F}'(0) \\ &= (3, 8) + (7, 1)t \\ &= (3 + 7t, 8 + t) \end{aligned}$$

Review: Curves and Tangent Lines

Example 2:

$$\mathbf{F}(x) = \left(x^3 + 7x + 3, 8 + \sin x, \ln(1 + x^2)\right), -2 \leq x \leq 2.$$

$$\mathbf{F}(0) = (3, 8, 0)$$

So $\mathbf{F}'(x) = (3x^2 + 7, \cos x, \frac{2x}{1+x^2})$

Implying $\mathbf{F}'(0) = (7, 1, 0)$

Tangent Line :

$$\begin{aligned} \mathbf{L}(t) &= \mathbf{F}(0) + t\mathbf{F}'(0) \\ &= (3, 8, 0) + (7, 1, 0)t \\ &= (3 + 7t, 8 + t, 0) \end{aligned}$$

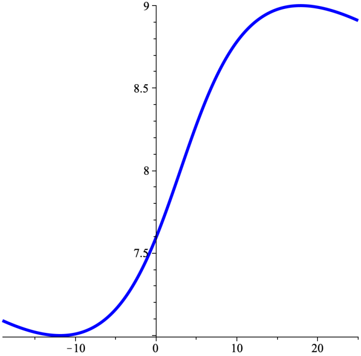
Plotting Vector-Valued Functions in *Maple*

$with(plots):$

Example of curve and tangent line

$\boldsymbol{F(x) = \left(x^3 + 7x + 3, 8 + \sin x\right)}$ ***with tangent line at*** $\boldsymbol{x = 0}$

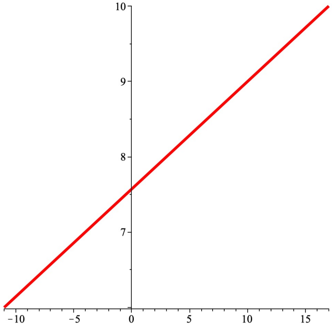
$Curve := plot\left(\left[x^3 + 7 \cdot x + 3, 8 + \sin(x)\right], x = -2..2\right], color = blue, thickness = 4\right)$



Plotting Vector-Valued Functions in Maple

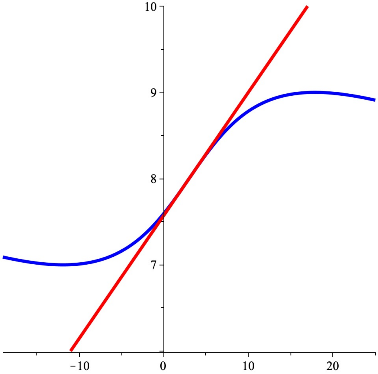
Equation of Tangent Line : $L(t) = F(0) + t\,F'(0) = (3, 8) + t(7, 1) = (3 + 7t, 8 + t)$

$Tangent := plot([3 + 7 \cdot t, 8 + t, t = -2..2], color = red, thickness = 4)$



Plotting Vector-Valued Functions in Maple

display(*Curve*, *Tangent*)

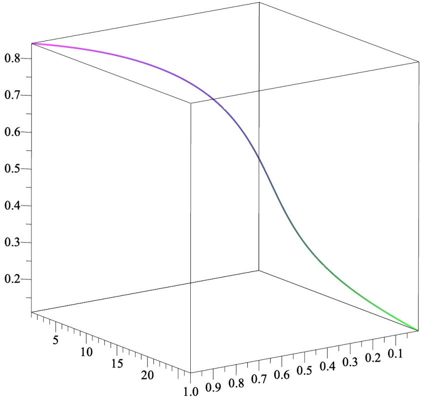


Plotting Vector-Valued Functions in Maple

Plotting a Curve in 3-space:

$with(plots):$

$$spacecurve\left(\left[\, t^7, t^{-3}, \sin\left(t^2\right)\right], t = \frac{1}{3}..1, thickness = 4\right)$$



Integrating Vector-Valued Functions of a Real Number

Example 3: Find all vector-valued functions of a real variable $\mathbf{p}(t)$ such that

$$\mathbf{p}'(t) = \left(\frac{1}{t^2+1}, \frac{t}{t^2+1} \right)$$

Solution:

$$\begin{aligned}
\mathbf{p}(t) = \int \mathbf{p}'(t)dt &= \int \left(\frac{1}{t^2+1}, \frac{t}{t^2+1} \right) dt \\
&= \left(\arctan t, \frac{1}{2} \ln \left(t^2+1 \right) \right) + (C_1, C_2) \\
&= \left(\arctan t + C_1, \frac{1}{2} \ln \left(t^2+1 \right) + C_2 \right)
\end{aligned}$$

Example 3 Continued: $\mathbf{p}(t) = \left(\arctan t + C_1, \frac{1}{2}\ln\left(t^2+1\right) + C_2\right)$

(i) Find particular solution so that $\mathbf{p}(0) = (3, 4)$

$$3 = \arctan 0 + C_1 = 0 + C_1 \text{ so } C_1 = 3$$

$$4 = \frac{1}{2}\ln(1+0^2) + C_2 = 0 + C_2 \text{ so } C_2 = 4$$

$$\text{Hence } \mathbf{p}(t) = \left(\arctan t + 3\text{‘}, \frac{1}{2}\ln\left(t^2+1\right) + 4\right)$$

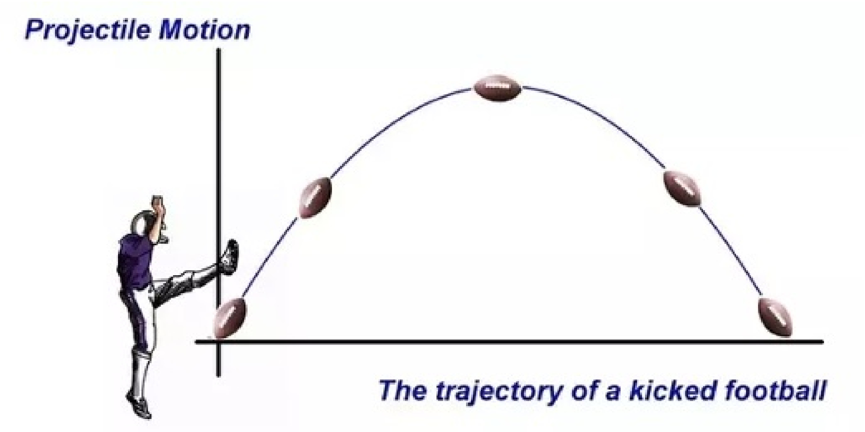
(ii) Find particular solution so that $\mathbf{p}(1) = (a, b)$

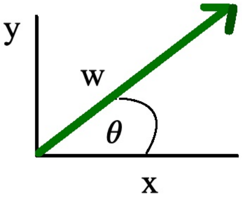
$$a = \arctan 1 + C_1 = \pi/4 + C_1 \text{ so } C_1 = a - \pi/4$$

$$b = \frac{1}{2}\ln(1+1^2) + C_2 = \frac{1}{2}\ln 2 + C_2 \text{ so } C_2 = b - \frac{1}{2}\ln 2$$

$$\text{Hence } \mathbf{p}(t) = \left(\arctan t + a - \pi/4\text{‘}, \frac{1}{2}\ln\left(t^2+1\right) + b - \frac{1}{2}\ln 2\right)$$

Projectile Motion


Projectile Motion
The trajectory of a kicked football



Initial Velocities: $x_0' = w\cos\theta, y_0' = w\sin\theta$

x and y are functions of time t

$$x'' = 0$$
$$x' = C = w\cos\theta$$
$$x = w\cos\theta\ t + x_0$$

$$y'' = -g$$
$$y' = -gt + C = -gt + w\sin\theta$$
$$y = -\frac{g}{2}t^2 + w\sin\theta\ t + y_0$$

$$x(t) = w\cos\theta\ t + x_0,\ y(t) = -\frac{g}{2}t^2 + w\sin\theta\ t + y_0$$

$$x(t) = w\cos\theta\ t + x_0$$
$$y(t) = -\tfrac{g}{2}t^2 + w\sin\theta\ t + y_0$$

Suppose $x_0 = 0, y_0 = 0$

Then

$$x(t) = w\cos\theta\ t$$
$$y(t) = -\tfrac{g}{2}t^2 + w\sin\theta\ t$$

Note:

$$t = \frac{x}{w\cos\theta}$$
$$y = -\frac{g}{2}\left(\frac{x^2}{w^2\cos^2\theta}\right) + \frac{w\sin\theta}{w\cos\theta}\,x$$

Graph of y versus x is a Downward Pointing Parabola

$$x(t) = w\cos\theta\ t$$
$$y(t) = -\tfrac{g}{2}t^2 + w\sin\theta\ t = t\left(-\tfrac{g}{2}t + w\sin\theta\right)$$

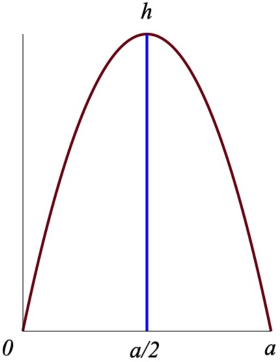
Now $y = 0$ at $t = 0$ and when $w\sin\theta = \frac{g}{2}$

which occurs when $t = \frac{2w\sin\theta}{g}$

At this time

$$x = w\cos\theta\ \left(\frac{2w\sin\theta}{g}\right) = \frac{2w^2\sin\theta\cos\theta}{g} = \frac{w^2}{g}\sin(2\theta)$$

So Maximum Horizontal Distance occurs when $sin(2\theta) = 1$;
that is, $\theta = \frac{\pi}{4}$



Maximum height h

$$\frac{a}{2} = w\cos\theta\ t,\ h = -\frac{g}{2}t^2 + w\sin\theta\ t$$

Maximum h occurs when $t = \frac{w\sin\theta}{g}$ [where $h'(t) = 0$]

At this time, $x = w\cos\theta\frac{w\sin\theta}{g} = \frac{w^2}{g}\sin\theta\cos\theta$

$$h = -\frac{g}{2} t^2 + w \sin\theta \; t$$

At $t = \frac{w \sin\theta}{g}$:

$$\begin{aligned} h &= -\frac{g}{2}\left(\frac{w \sin\theta}{g}\right)^2 + w \sin\theta \left(\frac{w \sin\theta}{g}\right) \\ &= -\frac{g}{2}\frac{w^2 \sin^2\theta}{g^2} + \frac{w^2 \sin^2\theta}{g} \\ &= -\frac{w^2 \sin^2\theta}{2g} + \frac{w^2 \sin^2\theta}{g} \\ &= \frac{w^2 \sin^2\theta}{2g} \end{aligned}$$

where $x = \frac{w^2}{g} \sin\theta \cos\theta$

To have $y = h$ at $x = a/2$:

$$h = \frac{w^2 \sin^2\theta}{2g}, \ \frac{a}{2} = \frac{w^2}{g}\sin\theta\cos\theta$$

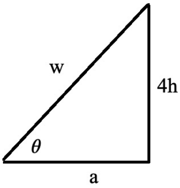
$$\frac{h}{a/2} = \left(\frac{w^2 \sin^2\theta}{2g}\right)\left(\frac{g}{w^2 \sin\theta\cos\theta}\right)$$

$$\frac{2h}{a} = \frac{\sin\theta}{2\cos\theta}$$

$$\frac{4h}{a} = \tan\theta$$

$$\theta = \arctan\left(\frac{4h}{a}\right)$$

$$\text{Recall } h = \frac{w^2 \sin^2\theta}{2g} \text{ and } \tan\theta = \frac{4h}{a}$$



$$w = \sqrt{16h^2 + a^2} \text{ so } \sin\theta = \frac{4h}{\sqrt{16h^2 + a^2}}$$

$$h = \frac{w^2}{2g}\sin^2\theta = \frac{w^2}{2g}\left(\frac{16h^2}{16h^2 + a^2}\right)$$

Solve for w^2:

$$w^2 = \frac{(2gh)(16h^2 + a^2)}{16h^2} = \frac{g(16h^2 + a^2)}{8h}$$

Real-Valued Functions of Vectors

Example: $f : \mathbb{R}^2 \to \mathbb{R}^1$

$$f(x,y) = \begin{cases} \frac{xy}{x^2+2y^2} & (x,y) \neq (0,0) \\ 0 & (0,0) \end{cases}$$

What happens as (x, y) approaches (0,0)?
Consider approaching along line $y = mx$. Then

$$\frac{xy}{x^2+2y^2} = \frac{xmx}{x^2+2(m^2x^2)} = \frac{m}{1+2m^2}$$