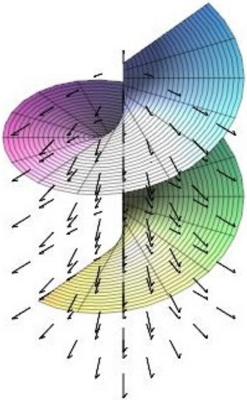
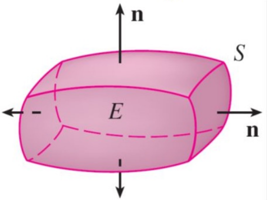
MATH 223: Multivariable Calculus

Class 35: December 9, 2022



Divergence Theorem

$$\iiint_E \operatorname{div} \mathbf{F}\, dV = \iint_S \mathbf{F} \cdot d\mathbf{S}$$



Notes on Assignment 32

Assignment 33

Stokes Theorem

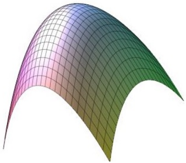
Announcements

Independent Project Due Tomorrow/Monday

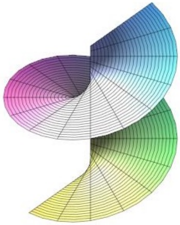
Course Response Forms
In Class Next Monday
Bring Laptop/SmartPhone

Final Exam
Wednesday, December 14: 9 AM – Noon

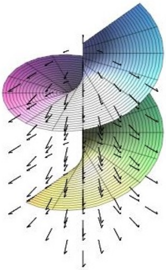
Integrating Vector Fields Over Surfaces



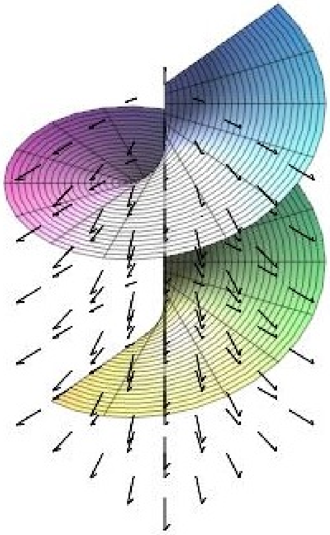
$g(u,v) = [u, v, -2u^2 - 3v^2]$



$g(u,v) = [u\cos v, u\sin v, v]$



Integrating Vector Fields Over Surfaces



$$g(u, v) = [u \cos v, u \sin v, v]$$

Smooth Curve γ	Smooth Surface S
$g : I$ in $\mathbb{R}^1 \to \mathbb{R}^n$	$g : D$ in $\mathbb{R}^2 \to \mathbb{R}^3$
Length $= \int_I \lvert g'(t)\rvert \, dt$	Area $\sigma(S) = \iint_D \lvert g_u \times g_v\rvert du dv$
Mass $= \int_I \mu(g(t))\lvert g'(t)\rvert \, dt$	Mass $= \iint_D \mu \, d\sigma$
Line Integral $\int_\gamma \mathbf{F} = \int_I \mathbf{F}(g(t)) \cdot g'(t) \, dt$	Surface Integral $\iint_S \mathbf{F} = \iint_D \mathbf{F}(g(u,v)) \cdot (g_u \times g_v)$

$$\iint_S \mathbf{F} = \iint_S \mathbf{F} \cdot dS = \iint_S \mathbf{F} \cdot \mathbf{N} \, d\sigma$$

$\Phi(\mathbf{F}, S) = \iint_S \mathbf{F}$ is **flux** of $\mathbf{F}$ across S.

Surface Integral

Let g be a function from an interval $[t_0, t_1]$ into $\mathbb{R}^n$ with image γ and μ density at $g(t)$.

Then Mass of Wire $= \int_{t_0}^{t_1} \mu(t)|g'(t)|\, dt$

If $\mu \equiv 1$, then mass $=$ length of curve $\int_{t_0}^{t_1} |g'(t)|\, dt$

Generalize To Surfaces

Let D be region in plane and $g : D \to \mathbb{R}^3$ with $g(u, v) = (g_1, g_2, g_3)$ where each component function g_i is continuously differentiable.

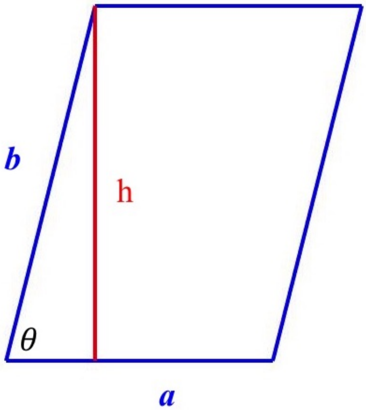
There are two natural tangent vectors: $g_u = \frac{\partial g}{\partial u}$ and $g_v = \frac{\partial g}{\partial v}$, These determine a tangent plane.

S is a **Smooth Surface** if these two vectors are linearly independent.

Note that $\frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v}$ is normal to the plane with

$$|\frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v}| = |\frac{\partial g}{\partial u}||\frac{\partial g}{\partial v}| \sin\theta$$

$=$ Area of Parallelogram Spanned by the Vectors



$$\sin\theta = \frac{h}{|\mathbf{b}|} \text{ so } h = |\mathbf{b}|\sin\theta$$

$$\text{Area of Parallelogram} = (\text{Base})(\text{Height}) = |\mathbf{a}||\mathbf{b}|\sin\theta$$

$$\mathbf{a} = g_u, \mathbf{b} = g_v$$

$$|g_u \times g_v| = |g_u||g_v|\sin\theta$$

Surface Area

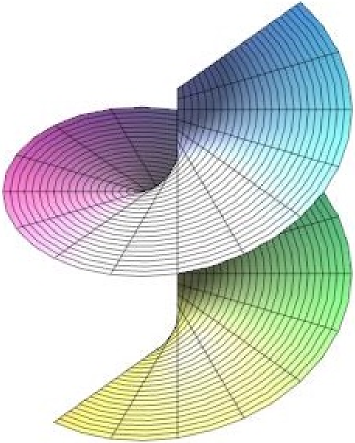
$$\sigma(S) = \iint_D |\frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v}| \, dudv = \iint_D |g_u \times g_v| \, dudv$$

If $\mu(g(u,v))$ is density, then mass $=$
$\iint_D \mu \, d\sigma = \iint_D \mu(g(u,v))|g_u \times g_v| \, dudv$

Plotting Parametrized Surface in $Maple$:
$plot3d([g1(u,v), g2(u,v), g3(u,v)], u = ..., v = ...)$

Area of a Spiral Ramp

$$g(u, v) = (u \cos v, u \sin v, v), 0 \le u \le 1, 0 \le v \le 3\pi$$



Area of a Spiral Ramp

$$g(u,v) = (u\cos v, u\sin v, v), 0 \le u \le 1, 0 \le v \le 3\pi$$

$$g_u = (\cos v, \sin v, 0), g_v = (-u\sin v, u\cos v, 1)$$

$$g_u \times g_v = \text{det} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 1 \end{vmatrix}$$

$$= \left(\begin{vmatrix} \sin v & 0 \\ u\cos v & 1 \end{vmatrix}, - \begin{vmatrix} \cos v & 0 \\ -u\sin v & 1 \end{vmatrix}, \begin{vmatrix} \cos v & \sin v \\ -u\sin v & u\cos v \end{vmatrix} \right)$$

$$= (\sin v, -\cos v, u)$$

Then $|g_u \times g_v| = \sqrt{\sin^2 v + \cos^2 v + u^2} = \sqrt{1+u^2}$

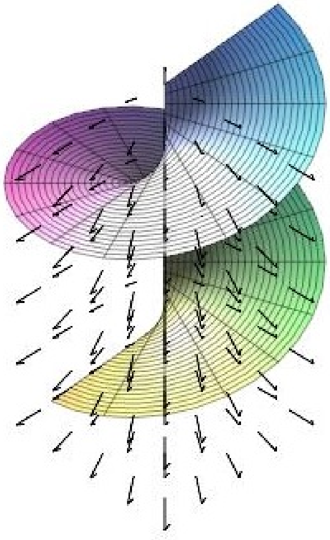
Area $= \int_{v=0}^{v=3\pi} \int_{u=0}^{1} \sqrt{1+u^2}\, du\, dv$

If density is $\mu(\mathbf{x}) = u$, then Mass $=$

$$\int_{v=0}^{v=3\pi} \int_{u=0}^{u=1} u(1+u^2)^{1/2}\, du\, dv = \int_{v=0}^{v=3\pi} \left[\tfrac{1}{3}(1+u^2)^{3/2}\right]_0^1 dv$$

$$= \int_{v=0}^{v=3\pi} \tfrac{1}{3}[2^{3/2} - 1^{3/2}]\, dv = 3\pi\tfrac{1}{3}[2^{3/2} - 1] = \pi[2^{3/2} - 1]$$

Integrating A Vector Field Over the Spiral Ramp



Integrating A Vector Field Over the Spiral Ramp

$$g(u,v) = (u\cos v, u\sin v, v), 0 \le u \le 1, 0 \le v \le 3\pi$$

$$g_u = (\cos v, \sin v, 0), g_v = (-u\sin v, u\cos v, 1)$$

$$g_u \times g_v = (\sin v, -\cos v, u)$$

Suppose our vector field is $\mathbf{F}(x,y,z) = (x^2, 0, z^2)$

So $F(g(u,v)) = (u^2\cos^2 v, 0, v^2)$

The set $D = \{(u,v) : 0 \le u \le 1, 0 \le v \le 3\pi\}$

We want $\int_D F(g(u,v)) \cdot (g_u \times g_v)$

which equals $\int_{v=0}^{3\pi} \int_{u=0}^{1} \left[u^2\cos^2 v \sin v + uv^2\right]\, du\, dv$

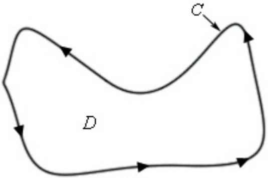
$$= \int_{v=0}^{3\pi} \left[\frac{u^3}{3}\cos^2 v\sin v + \frac{u^2}{2}v^2\Big|_{u=0}^{1}\right]\, dv =$$

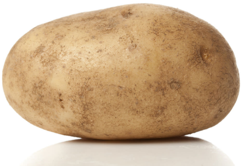
$$\int_{v=0}^{3\pi} \left[\frac{1}{3}\cos^2 v\sin v + \frac{1}{2}v^2\right] dv$$

$$= \left[\frac{-\cos^3 v}{9} + \frac{v^3}{6}\right]_{v=0}^{3\pi} = \frac{1}{9} + \frac{3^3 Pi^3}{6} - \frac{-1}{9} = \frac{2}{9} + \frac{9}{2}\pi^3$$

Gauss's Theorem aka Divergence Theorem

Planar Version: $\int_D \text{div } \mathbf{F} = \int_\gamma \mathbf{F} \cdot \mathbf{N}$





Three Dimensional Version

∂R is 2-dimensional surface surrounding 3-dimensional region R

$\int_R \text{div } \mathbf{F} = \int_{\partial R} \mathbf{F} \cdot \mathbf{N}$

Gauss's Theorem

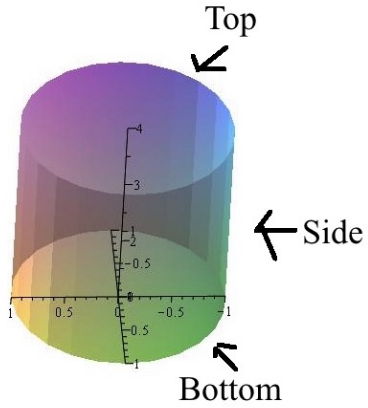
The Setting

$\mathcal{R}$	Bounded Solid Region in $\mathbb{R}^3$
$\partial\mathcal{R}$	Finitely Many Piecewise Smooth, Closed Orientable Surfaces Oriented by Unit Normals Pointed away from $\mathcal{R}$
$\mathbf{F}$	Continuously Differentiable Vector Field in $\mathcal{R}$

The Theorem

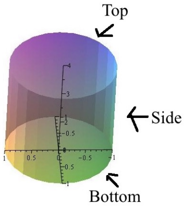
In this setting $\displaystyle\int_{\mathcal{R}} \text{div } \mathbf{F}\, dV = \int_{\partial\mathcal{R}} \mathbf{F} \cdot d\mathbf{S}$

Example Verify Gauss's Theorem where $\mathcal{R}$ is solid cylinder of radius a and height b with the z-axis as the axis of the cylinder and $\mathbf{F} = (x, y, z)$



$$\int_S \mathbf{F} \cdot dS = \int_{Bottom} \mathbf{F} \cdot dS + \int_{Top} \mathbf{F} \cdot dS + \int_{Side} \mathbf{F} \cdot dS$$

Cylinder of Radius a and height b

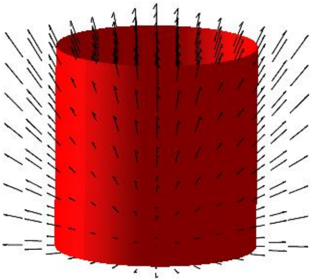


$$\int_S \mathbf{F} \cdot dS = \int_{Bottom} \mathbf{F} \cdot dS + \int_{Top} \mathbf{F} \cdot dS + \int_{Side} \mathbf{F} \cdot dS$$

For $\int_{Bottom} \mathbf{F} \cdot dS$, unit normal is (0,0,-1)
Then $(x, y, z) \cdot (0, 0, -1) = -z$ but $z = 0$ so $\int_{Bottom} \mathbf{F} \cdot dS = 0$

For $\int_{Top} \mathbf{F} \cdot dS$, unit normal is (0,0,1)
Then $(x, y, z) \cdot (0, 0, +1) = z$ but $z = b$ so $\int_{Top} \mathbf{F} \cdot dS$
is $b \times$ area of top $= b\pi a^2$

Finally, $\int_{Side} \mathbf{F} \cdot dS$



Vector Field $\mathbf{F} = (x, y, z)$

Surface: $x^2 + y^2 = a^2, 0 \leq z \leq b$

$$g(u, v) = (a\cos u, a\sin u, v), 0 \leq u \leq 2\pi, 0 \leq v \leq b$$

Finally, $\int_{Side} \mathbf{F} \cdot dS$

$g(u,v) = (a\cos u, a\sin u, v), 0 \le u \le 2\pi, 0 \le v \le b$

$g_u = (-a\sin u, a\cos u, 0), \quad g_v = (0,0,1)$

$$g_u \times g_v = \text{ det } \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin u & a\cos u & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

=(expanding along bottom row) $(a\cos u, a\sin u, 0)$

Thus $|g_u \times g_v| = \sqrt{a^2\cos^2 u + a^2\sin^2 u + 0^2} = a$

Also $F(g(u,v)) = (a\cos u, a\sin u, v)$ so $F(g(u,v)) \cdot (g_u \times g_v) = a^2\cos^2 u + a^2\sin^2 u + 0 = a^2$.

so $\int_{Side} \mathbf{F} \cdot dS = \int_{v=0}^{b} \int_{u=0}^{2\pi} a^2 \, du \, dv = 2\pi a^2 b$

Putting it altogether: $\int_S \mathbf{F} \cdot dS$

$= \int_{Bottom} \mathbf{F} \cdot dS + \int_{Top} \mathbf{F} \cdot dS + \int_{Side} \mathbf{F} \cdot dS = 0 + \pi a^2 b + 2\pi a^2 b = 3\pi a^2 b$

On The Other Hand, we compute $\int_R$ div **F**

$\mathbf{F} = (x, y, z)$

div $\mathbf{F} = 1 + 1 + 1 = 3$

The solid R is more easily described in polar coordinates

$0 \le \theta \le 2\pi \quad 0 \le r \le a \quad 0 \le z \le b.$

$$\int_R \text{div } \mathbf{F} = \int_{\theta=0}^{2\pi} \int_{z=0}^{b} \int_{r=0}^{a} \text{div } \mathbf{F} r dr dz d\theta = \int_{\theta=0}^{2\pi} \int_{z=0}^{b} \int_{r=0}^{a} 3r dr dz d\theta$$

$$\int_{\theta=0}^{2\pi} \int_{z=0}^{b} 3\frac{r^2}{2}|_{r=0}^{a} dz d\theta = \int_{\theta=0}^{2\pi} \int_{z=0}^{b} \frac{3}{2}a^2 dz d\theta = \int_{\theta=0}^{2\pi} \frac{3}{2}a^2 b d\theta = 2\pi\frac{3}{2}a^2 b$$

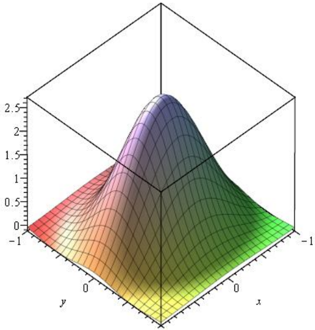
$$= 3a^2 b\pi$$

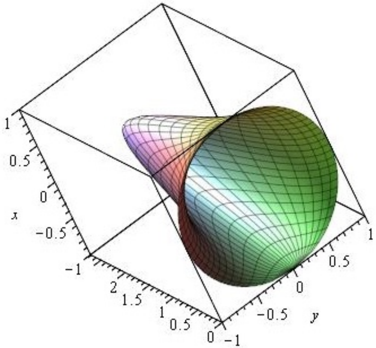
Example: $\mathbf{F} = (e^y \cos z, \sqrt{x^3+1}\sin z, x^2+y^2+3)$

$\text{div } \mathbf{F} = 0 + 0 + 0 = 0$

so $\int_R \text{div } \mathbf{F} = 0$ for any region in $\mathbb{R}^3$.

Let S be graph of $z = (1-x^2-y^2)e^{1-x^2-3y^2})$ for $z \geq 0$ oriented by outward pointing unit normal vector.





Finding $\int_S \mathbf{F} \cdot d\sigma$ directly is impossible.

A Clever Way To Find $\int_S \mathbf{F} \cdot d\sigma$ indirectly.

Cap the Surface with a Disk so New Surface Bounds a 3-Dimensional Region

Form closed surface $S \cup S'$ where S' is the disk of radius 1 $(x^2 + y^2 = 1)$ in $z = 0$ plane.

Then $\int_{\partial r} \mathbf{F} = \int_{S \cup S'} \mathbf{F} = \int_S \mathbf{F} + \int_{S'} \mathbf{F}$
But by Gauss's Theorem, this integral equals 0.
Hence $\int_S \mathbf{F} = -\int_{S'} \mathbf{F}$

Now

$$\int_{S'} \mathbf{F} = -\int (--, --, x^2 + y^2 + 3) \cdot (0, 0, -1) = \int x^2 + y^2 + 3\, dx\, dy$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} (r^2 + 3)\, r\, dt\, d\theta = \frac{7}{2}\pi$$

Next Time:

Stokes's Theorem

$$\int_S \text{curl } \mathbf{F} = \int_{\partial S} \mathbf{F}$$

S is a Surface in $\mathbb{R}^3$

Theorem: A continuously differentiable gradient field has a symmetric Jacobian matrix.

Proof: If $\mathbf{F}$ is a gradient field, then $\mathbf{F} = \nabla f$ for some real-valued function f.

Then $\mathbf{F} = (f_x, f_y)$ so the Jacobian matrix is

$$J = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

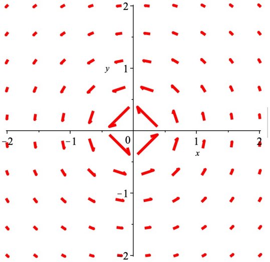
By Continuity of Mixed Partials, $f_{xy} = f_{yx}$ so J is symmetric. □

Theorem: If $\mathbf{F}$ is conservative, then its Jacobian is symmetric.

Theorem: If $\mathbf{F}$ is conservative, then its Jacobian is symmetric.

The converse (Symmetric Jacobian Implies Conservative) is **FALSE** in general.

Example: Consider the vector field $\mathbf{F}(x, y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$



defined for all $(x, y) \neq (0, 0)$

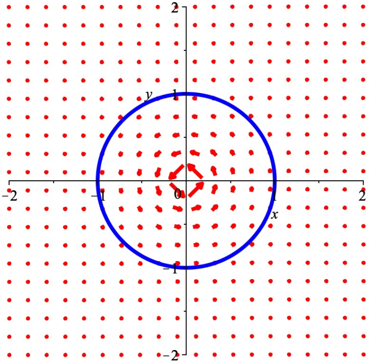
Then Jacobian $= \begin{pmatrix} - & \frac{y^2-x^2}{(x^2+y^2)^2} \\ \frac{y^2-x^2}{(x^2+y^2)^2} & - \end{pmatrix}$

$$\mathbf{F}(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$$

Has a Symmetric Jacobian But Is Not Conservative!

If $\mathbf{F}$ were conservative, then the line integral of $\mathbf{F}$ around any closed loop would be 0.

Consider γ the unit circle as a loop running counterclockwise starting and ending at (1.0).



$$\mathbf{F}(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$$

γ: unit circle as a loop running counterclockwise starting and ending at (1.0).
We parametrize γ by $g(t) = (\cos t, \sin t), 0\pi$ so that $g'(t) = (-\sin t, \cos t)$ and

$$\mathbf{F}(g(t)) = \left(\frac{-\sin t}{\cos^2 t + \sin^2 t}, \frac{\cos t}{\cos^2 t + \sin^2 t}\right) = (-\sin t, \cos t)$$

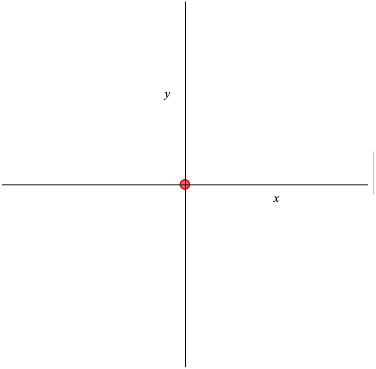
$$\mathbf{F}(g(t)) \cdot g'(t) = (-\sin t, \cos t) \cdot (-\sin t, \cos t) = \sin^2 t + \cos^2 t = 1$$

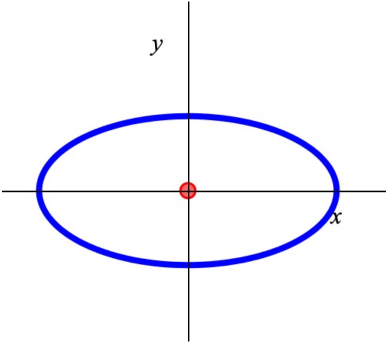
Thus $\int_\gamma \mathbf{F} = \int_0^{2\pi} 1\, dt = 2\pi \neq 0.$

What is Wrong the Vector Field

$$\mathbf{F}(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)?$$

The Domain of the Vector Field
(Plane minus the Origin)
Is Not Simply Connected.



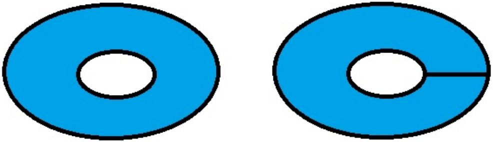


Simple Connectedness

A set B is **simply connected** if every closed curve in B can be continuously contracted to a point in such a way as to stay in B during the contraction. More precisely,

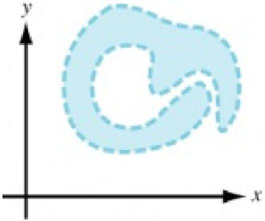
Definition: An open set B is**simply connected** if every piecewise smooth closed curve lying in B is the border of some piecewise smooth orientable surface S lying in B, and with parameter domain a disk in $\mathcal{R}^2$.

Theorem: Let $\mathbf{F}$ be a continuously differentiable vector field defined on an open set B in $\mathcal{R}^2$ or $\mathcal{R}^3$. If B is simply connected and curl F is identically zero in B, then $\mathbf{F}$ is a gradient field in B ; that is, there is a real-valued function f such that $\mathbf{F} = \nabla f$

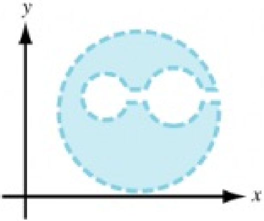


not simply connected

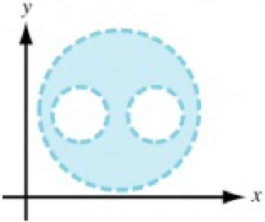
simply connected
thanks to single cut

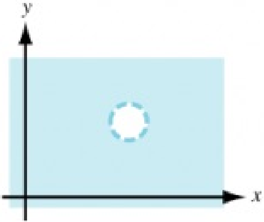


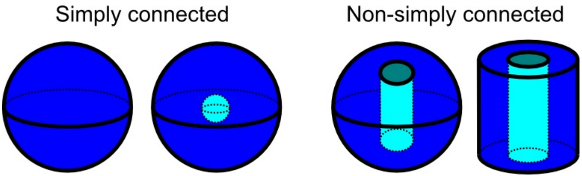
(a) A simply connected domain

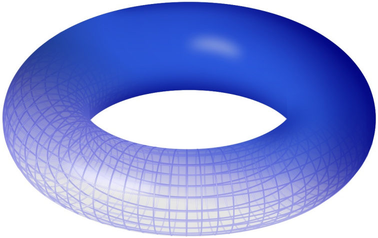


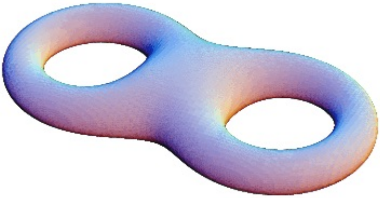
(b) A simply connected domain






Simply connected
Non-simply connected





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