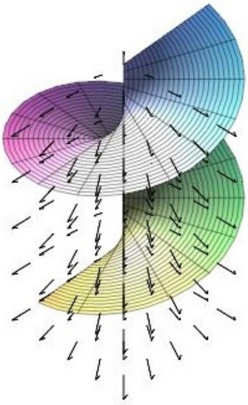
MATH 223: Multivariable Calculus



Class 34: December 7, 2022

Notes on Assignment 31

Assignment 32

Surface Integrals

Announcements

Course Response Forms

In Class, Monday, December 12

Bring Laptop/Smart Phone

Final Exam

Wednesday, December 14

9– Noon

Conservative Vector Fields

F is continuously differentiable vector field in the plane
F : $\mathbb{R}^2 \to \mathbb{R}^2$ with $\mathbf{F}(x,y) = (F(x,y), G(x,y))$ where F and G are each real-valued functions.

Here curl **F** is a real-valued function $G_x - F_y$

Green's Theorem: $\int_D \text{curl } \mathbf{F} = \int_\gamma \mathbf{F}$

Three Important Properties of Vector Fields

- **A** **F** is **CONSERVATIVE** means $\mathbf{F} = \nabla f$ for some $f : \mathbb{R}^2 \to \mathbb{R}^1$
- **B** **F** is **IRROTATIONAL** means curl $\mathbf{F} = 0$
- **C** **F** is **PATH INDEPENDENT** means $\int_{\gamma_1} \mathbf{F} = \int_{\gamma_2} \mathbf{F}$ for any paths γ_1 and γ_2 from **a** to **b** where **a** and **b** are any points in the plane.

Major Goal: Show THESE PROPERTIES ARE EQUIVALENT

A implies **B**

A **F** is **CONSERVATIVE** means **F** $= \nabla f$ for some $f : \mathbb{R}^2 \to \mathbb{R}^1$

B **F** is **IRROTATIONAL** means curl **F** $= 0$

Suppose **F** is Conservative

Then $(F, G) =$ **F** $= \nabla f = (f_x, f_y)$ so $f_x = F$ and $f_y = G$

Thus $G_x = f_{yx}$ and $F_y = f_{xy}$

so curl **F** $= G_x - F_y = f_{yx} - f_{xy} = 0$

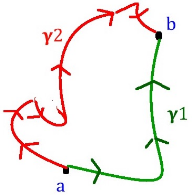
by equality of mixed partials.

B implies **C** will follow from Green's Theorem

B **F** is **IRROTATIONAL** means curl **F** $= 0$

C **F** is **PATH INDEPENDENT** means $\int_{\gamma_1} \mathbf{F} = \int_{\gamma_2} \mathbf{F}$ for any paths γ_1 and γ_2 from **a** to **b** where **a** and **b** are any points in the plane.

Let **a** and **b** are any points in the plane and γ_1 and γ_2 two paths from **a** to **b**. Then $-\gamma_1$ runs from **b** to **a**



and $\gamma = \gamma_1 - \gamma_2$ is a loop that begins and ends at **a**

Let D be the enclosed region.

By Green's Theorem $\int_\gamma \mathbf{F} = \iint_D$ curl $\mathbf{F} = \iint_D 0 = 0$

Thus $0 = \int_\gamma \mathbf{F} = \int_{\gamma_1 - \gamma_2} \mathbf{F} = \int_{\gamma_1} \mathbf{F} - \int_{\gamma_2} \mathbf{F}$

Hence $\int_{\gamma_2} \mathbf{F} = \int_{\gamma_1} \mathbf{F}$

C implies **A**

C **F** is **PATH INDEPENDENT** means $\int_{\gamma_1} \mathbf{F} = \int_{\gamma_2} \mathbf{F}$ for any paths γ_1 and γ_2 from **a** to **b** where **a** and **b** are any points in the plane.

A **F** is **CONSERVATIVE**means $\mathbf{F} = \nabla f$ for some $f : \mathbb{R}^2 \to \mathbb{R}^1$

Idea:
Fix $\mathbf{x}_0$ in $\mathbb{R}^n$ and let $\mathbf{x}$ be arbitrary point in $\mathbb{R}^n$.
Let γ be a curve from $\mathbf{x}_0$ to $\mathbf{x}$.
Then $\int_\gamma \mathbf{F}$ will be a function of $\mathbf{x}$ whose gradient is $\mathbf{F}$.

<u>Theorem</u> Let **F** be a continuous vector field defined in a polygonally connected open set D of $\mathbb{R}^n$. If the line integral $\int_\gamma \mathbf{F}$ is independent of piecewise smooth path γ from $\mathbf{x}_0$ to $\mathbf{x}$ in D, then if $f(\mathbf{x}) = \int_\gamma \mathbf{F}$, it is true that $\nabla f = \mathbf{F}$.

Example $\mathbf{F}(x,y) = (3x^2 + y, e^y + x)$
Here $\mathbf{F} = (F, G)$ so $F(x,y) = 3x^2 + y, G(x,y) = e^y + x$
Hence $F_y = 1, G_x = 1$ so curl $\mathbf{F} = G_x - F_y = 0$
Let's build f so its gradient $\nabla f = (f_x, f_y) = (3x^2 + y, e^y + x)$
We need $f_x = 3x^2 + y$ so do "partial integration with respect to x":
$f(x) = x^3 + yx + g(y)$. [Why is there $g(y)$?]
Then $f_y = 0 + x + g'(y)$ which should equal $x + e^y$ so need

$$g'(y) = e^y$$

which we can get by letting $g(y) = e^y$.
Hence we can choose $f(x,y) = x^3 + yx + e^y + C$.

Let's build the potential function in a different way using the theorem with $\mathbf{F}(x, y) = (3x^2 + y, e^y + x)$

Pick $\mathbf{x}_0 = (0, 0)$ and let $\mathbf{x} = (x, y)$ be an arbitrary point. Choose the straight line between them as the path γ with parametrization

$g(t) = (xt, yt), 0 \leq t \leq 1$ so $g'(t) = (x, y)$

Then $\mathbf{F}(g(t)) = F(xt, yt) = (3x^2t^2 + yt, e^{yt} + x)t$

so $\mathbf{F}(g(t)) \cdot g'(t) = (3x^2t^2 + yt, e^{yt} + xt) \cdot (x, y)$

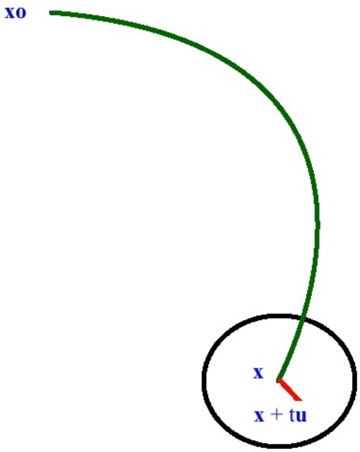
$= 3x^3t^2 + xyt + ye^{yt} + xyt = 3x^3t^2 + 2xyt + ye^{yt}$

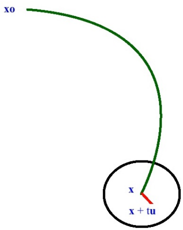
Now $\int_\gamma \mathbf{F} = \int_0^1 (3x^3t^2 + 2xyt + ye^{yt})\, dt$

$= \left[x^3t^3 + xyt^2 + e^{yt}\right]_{t=0}^{t=1}$

$= (x^3 + xy + e^y) - (0 + 0 + 1) = x^3 + xy + e^y - 1$

Theorem Let $\mathbf{F}$ be a continuous vector field defined in a polygonally connected open set D of $\mathbb{R}^n$. If the line integral $\int_\gamma \mathbf{F}$ is independent of piecewise smooth path γ from $\mathbf{x}_0$ to $\mathbf{x}$ in D, then if $f(\mathbf{x}) = \int_\gamma \mathbf{F}$, it is true that $\nabla f = \mathbf{F}$.





Let g be a parametrization of line segment from $\mathbf{x}$ to $\mathbf{x}+t\mathbf{u}$ so $g(t)=\mathbf{x}+v\mathbf{u}, 0 \le v \le t$ and $g'(t)=\mathbf{u}$

$$f(\mathbf{x}+t\mathbf{u})-f(\mathbf{x}) = \int_{\mathbf{x}_0}^{\mathbf{x}+t\mathbf{u}} \mathbf{F} - \int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{F} = \int_{\mathbf{x}}^{\mathbf{x}+t\mathbf{u}} \mathbf{F}(\mathbf{x}+v\mathbf{u})$$
$$= \int_0^t \mathbf{F}(\mathbf{x}+v\mathbf{u}) \cdot \mathbf{u}\, dv$$

To find $\frac{\partial f}{\partial x_j}(\mathbf{x})$, let $\mathbf{u}$ be the unit vector $\mathbf{e}_j = (0, 0, \ldots, 1, 0, 0. \ldots 0)$ in the jth direction.

$$\frac{\partial f}{\partial x_j}(\mathbf{x}) = \lim_{t \to 0} \frac{f(\mathbf{x} + t\mathbf{u}) - f(\mathbf{x})}{t}$$

$$= \lim_{t \to 0} \frac{1}{t} \int_0^t \mathbf{F}(\mathbf{x} + v\mathbf{u}) \cdot \mathbf{u} \, dv$$

$$= \lim_{t \to 0} \frac{1}{t} \int_0^t \mathbf{F}(\mathbf{x} + v\mathbf{e}_j) \cdot \mathbf{e}_j \, dv$$

But this last expression is the derivative of the integral with respect to t evaluated at $t = 0$ which is $\mathbf{F} \cdot \mathbf{e}_j = F_j(\mathbf{x})$ (Using Fundamental Theorem of Calculus)

Symmetry of Jacobian Matrix for Conservative Vector Field

Let **F** $= (F(x, y), G(x, y))$ be a conservative vector field in the plane which we can recognized by $G_x = F_y$

F' $= \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix}$ Note symmetry of Jacobian Matrix.

How do things generalize to higher dimensions?

Example: **F**: $\mathbb{R}^3 \to \mathbb{R}^3$ by

$$F(x,y,z) = (yz^2 + \sin y + 3x^2, xz^2 + x\cos y + e^z, 2xyz + ye^z + \frac{1}{z})$$

$$\mathbf{F'} = \begin{pmatrix} 6x & z^2+\cos y & 2yz \\ z^2+\cos y & -x\sin y & 2xz+e^z \\ 2yz & 2xz+e^z & 2xy+ye^z-\frac{1}{z^2} \end{pmatrix}$$

To find f so that $\nabla f = \mathbf{F}$:

Step 1: integrate first component of **F** with respect to x:

$$f(x,y,z) = yz^2x + x\sin y + x^3 + G(y,z)$$

Step 2: Take derivative of trial f respect to y and set equal to second component of **F** :

$f_y = z^2x + x\cos y + 0 + G_y(x,y)$ must $= xz^2 + x\cos y + e^z$

Need $G_y(x,y) = e^z$ so choose $G(x,y) = e^z y + H(z)$

So far, $f(x,y,z) = yz^2x + x\sin y + x^3 + e^z y + H(z)$

Step 3:Take derivative of trial f respect to z and set equal to third component of **F** ;

$f_z(x,y,z) = 2xyz + 0 + 0 + e^z y + H'(z)$ must $= 2xyz + e^z y + \frac{1}{z}$

Need $H'(z) = \frac{1}{z}$ so choose $H(x) = \ln|z| + C$

Thus

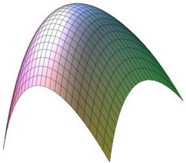
$$f(x,y,z) = f(x,y,z) = yz^2x + x\sin y + x^3 + e^z y + \ln|z| + C$$

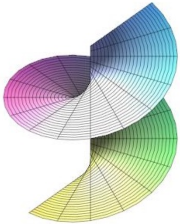
Theorem If **F** is a conservative vector field on $\mathbb{R}^N = n$ and is continuously differentiable, then the Jacobian matrix is symmetric.

Proof: Equality of mixed partials.

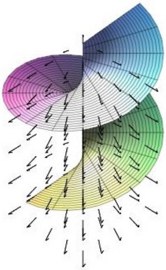
Theorem Suppose **F** is a continuously differentiable vector field on $\mathbb{R}^n$ whose Jacobian matrix is symmetric. Then **F** is conservative

Integrating Vector Fields Over Surfaces

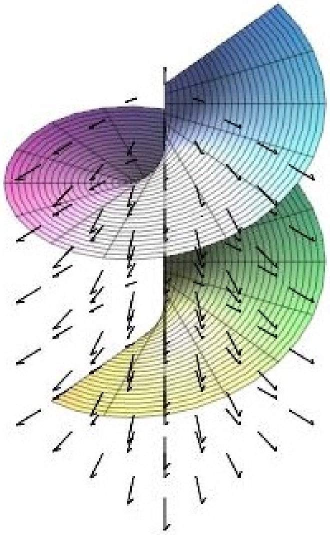




$g(u,v) = [u, v, -2u^2 - 3v^2]$ $\quad$ $g(u,v) = [u\cos v, u\sin v, v]$



Integrating Vector Fields Over Surfaces



$$g(u, v) = [u \cos v, u \sin v, v]$$

Smooth Curve γ	Smooth Surface S
$g : I$ in $\mathbb{R}^1 \to \mathbb{R}^n$	$g : D$ in $\mathbb{R}^2 \to \mathbb{R}^3$
Length $= \int_I \lvert g'(t)\rvert \, dt$	Area $\sigma(S) = \iint_D \lvert g_u \times g_v \rvert dudv$
Mass $= \int_I \mu(g(t))\lvert g'(t)\rvert \, dt$	Mass $= \iint_D \mu \, d\sigma$
Line Integral $\int_\gamma \mathbf{F} = \int_I \mathbf{F}(g(t)) \cdot g'(t) \, dt$	Surface Integral $\iint_S \mathbf{F} = \iint_D \mathbf{F}(g(u,v)) \cdot (g_u \times g_v)$

$$\iint_S \mathbf{F} = \iint_S \mathbf{F} \cdot dS = \iint_S \mathbf{F} \cdot \mathbf{N} \, d\sigma$$

$\Phi(\mathbf{F}, S) = \iint_S \mathbf{F}$ is **flux** of $\mathbf{F}$ across S.

Surface Integral

Let g be a function from an interval $[t_0, t_1]$ into $\mathbb{R}^n$ with image γ and μ density at $g(t)$.

Then Mass of Wire $= \int_{t_0}^{t_1} \mu(t)|g'(t)|\, dt$

If $\mu \equiv 1$, then mass $=$ length of curve $\int_{t_0}^{t_1} |g'(t)|\, dt$

Generalize To Surfaces

Let D be region in plane and $g : D \to \mathbb{R}^3$ with $g(u, v) = (g_1, g_2, g_3)$ where each component function g_i is continuously differentiable.

There are two natural tangent vectors: $g_u = \frac{\partial g}{\partial u}$ and $g_v = \frac{\partial g}{\partial v}$,

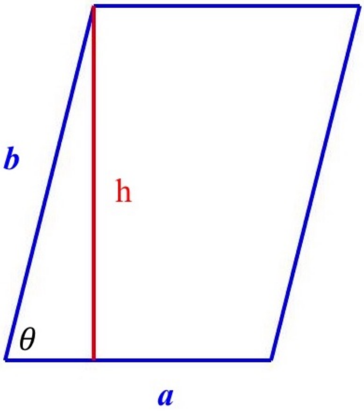
These determine a tangent plane.

S is a **Smooth Surface** if these two vectors are linearly independent.

Note that $\frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v}$ is normal to the plane with

$$|\frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v}| = |\frac{\partial g}{\partial u}||\frac{\partial g}{\partial v}| \sin\theta$$

$=$ Area of Parallelogram Spanned by the Vectors



$$\sin\theta = \frac{h}{|\mathbf{b}|} \text{ so } h = |\mathbf{b}|\sin\theta$$

$$\text{Area of Parallelogram} = \text{(Base)(Height)} = |\mathbf{a}||\mathbf{b}|\sin\theta$$

$$\mathbf{a} = g_u, \mathbf{b} = g_v$$

$$|g_u \times g_v| = |g_u||g_v|\sin\theta$$

Surface Area

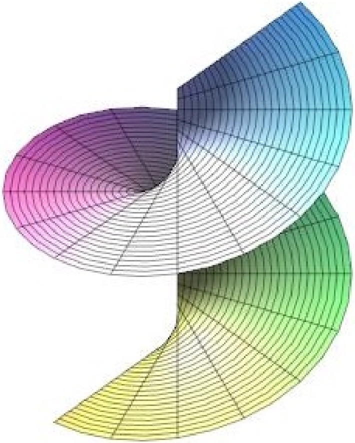
$$\sigma(S) = \iint_D |\frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v}|\, dudv = \iint_D |g_u \times g_v|\, dudv$$

If $\mu(g(u,v))$ is density, then mass $=$
$\iint_D \mu\, d\sigma = \iint_D \mu(g(u,v))|g_u \times g_v|\, dudv$

Plotting Parametrized Surface in $Maple$:
$plot3d([g1(u,v), g2(u,v), g3(u,v)], u = ..., v = ...)$

Area of a Spiral Ramp

$$g(u, v) = (u \cos v, u \sin v, v), 0 \leq u \leq 1, 0 \leq v \leq 3\pi$$



Area of a Spiral Ramp

$$g(u,v) = (u\cos v, u\sin v, v), 0 \le u \le 1, 0 \le v \le 3\pi$$

$$g_u = (\cos v, \sin v, 0), g_v = (-u\sin v, u\cos v, 1)$$

$$g_u \times g_v = \text{det} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 1 \end{vmatrix}$$

$$= \left(\begin{vmatrix} \sin v & 0 \\ u\cos v & 1 \end{vmatrix}, - \begin{vmatrix} \cos v & 0 \\ -u\sin v & 1 \end{vmatrix}, \begin{vmatrix} \cos v & \sin v \\ -u\sin v & u\cos v \end{vmatrix} \right)$$

$$= (\sin v, -\cos v, u)$$

Then $|g_u \times g_v| = \sqrt{\sin^2 v + \cos^2 v + u^2} = \sqrt{1+u^2}$

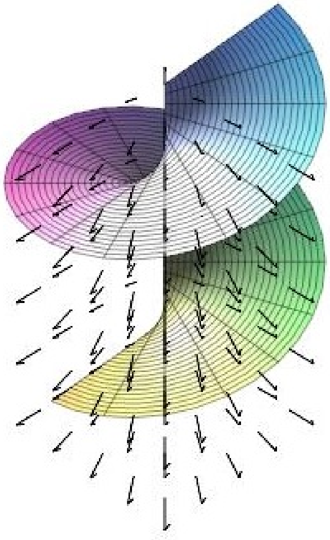
Area $= \int_{v=0}^{v=3\pi} \int_{u=0}^{1} \sqrt{1+u^2}\, du\, dv$

If density is $\mu(\mathbf{x}) = u$, then Mass $=$

$$\int_{v=0}^{v=3\pi} \int_{u=0}^{u=1} u(1+u^2)^{1/2}\, du\, dv = \int_{v=0}^{v=3\pi} \left[\tfrac{1}{3}(1+u^2)^{3/2}\right]_0^1 dv$$

$$= \int_{v=0}^{v=3\pi} \tfrac{1}{3}[2^{3/2} - 1^{3/2}]\, dv = 3\pi\tfrac{1}{3}[2^{3/2} - 1] = \pi[2^{3/2} - 1]$$

Integrating A Vector Field Over the Spiral Ramp



Integrating A Vector Field Over the Spiral Ramp

$$g(u,v) = (u\cos v, u\sin v, v), 0 \le u \le 1, 0 \le v \le 3\pi$$

$$g_u = (\cos v, \sin v, 0), g_v = (-u\sin v, u\cos v, 1)$$

$$g_u \times g_v = (\sin v, -\cos v, u)$$

Suppose our vector field is $\mathbf{F}(x,y,z) = (x^2, 0, z^2)$

So $F(g(u,v)) = (u^2\cos^2 v, 0, v^2)$

The set $D = \{(u,v) : 0 \le u \le 1, 0 \le v \le 3\pi\}$

We want $\int_D F(g(u,v)) \cdot (g_u \times g_v)$

which equals $\int_{v=0}^{3\pi} \int_{u=0}^{1} \left[u^2\cos^2 v \sin v + uv^2\right] \, du \, dv$

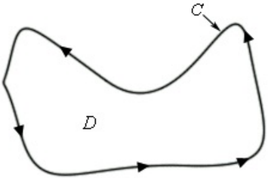
$$= \int_{v=0}^{3\pi} \left[\frac{u^3}{3}\cos^2 v \sin v + \frac{u^2}{2}v^2\Big|_{u=0}^{1}\right] \, dv =$$

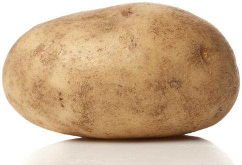
$$\int_{v=0}^{3\pi} \left[\tfrac{1}{3}\cos^2 v \sin v + \tfrac{1}{2}v^2\right] dv$$

$$= \left[\frac{-\cos^3 v}{9} + \frac{v^3}{6}\right]_{v=0}^{3\pi} = \frac{1}{9} + \frac{3^3 Pi^3}{6} - \frac{-1}{9} = \frac{2}{9} + \frac{9}{2}\pi^3$$

Gauss's Theorem aka Divergence Theorem

Planar Version: $\int_D \text{div } \mathbf{F} = \int_\gamma \mathbf{F} \cdot \mathbf{N}$





Three Dimensional Version

∂R is 2-dimensional surface surrounding 3-dimensional region R

$\int_R \text{div } \mathbf{F} = \int_{\partial R} \mathbf{F} \cdot \mathbf{N}$

Gauss's Theorem

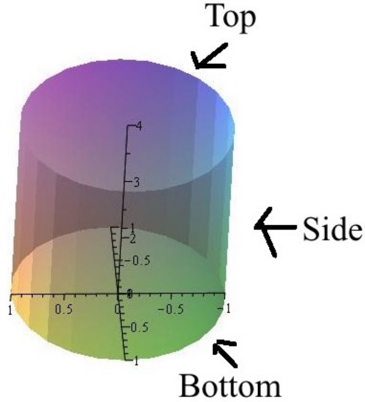
The Setting

$\mathcal{R}$	Bounded Solid Region in $\mathbb{R}^3$
$\partial\mathcal{R}$	Finitely Many Piecewise Smooth, Closed Orientable Surfaces Oriented by Unit Normals Pointed away from $\mathcal{R}$
$\mathbf{F}$	Continuously Differentiable Vector Field in $\mathcal{R}$

The Theorem

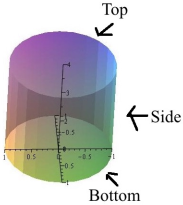
$$\text{In this setting } \int_{\mathcal{R}} \text{div } \mathbf{F} \, dV = \int_{\partial\mathcal{R}} \mathbf{F} \cdot d\mathbf{S}$$

Example Verify Gauss's Theorem where $\mathcal{R}$ is solid cylinder of radius a and height b with the z-axis as the axis of the cylinder and $\mathbf{F} = (x, y, z)$



$$\int_S \mathbf{F} \cdot dS = \int_{Bottom} \mathbf{F} \cdot dS + \int_{Top} \mathbf{F} \cdot dS + \int_{Side} \mathbf{F} \cdot dS$$

Cylinder of Radius a and height b

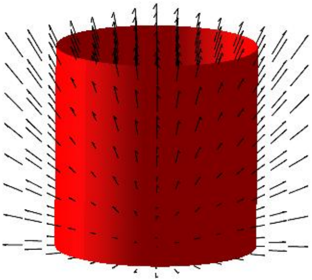


$$\int_S \mathbf{F} \cdot dS = \int_{Bottom} \mathbf{F} \cdot dS + \int_{Top} \mathbf{F} \cdot dS + \int_{Side} \mathbf{F} \cdot dS$$

For $\int_{Bottom} \mathbf{F} \cdot dS$, unit normal is (0,0,-1)
Then $(x, y, z) \cdot (0, 0, -1) = -z$ but $z = 0$ so $\int_{Bottom} \mathbf{F} \cdot dS = 0$

For $\int_{Top} \mathbf{F} \cdot dS$, unit normal is (0,0,1)
Then $(x, y, z) \cdot (0, 0, +1) = z$ but $z = b$ so $\int_{Top} \mathbf{F} \cdot dS$
is $b \times$ area of top $= b\pi a^2$

Finally, $\int_{Side} \mathbf{F} \cdot dS$



Vector Field $\mathbf{F} = (x, y, z)$

Surface: $x^2 + y^2 = a^2, 0 \leq z \leq b$

$g(u, v) = (a \cos u, a \sin u, v), 0 \leq u \leq 2\pi, 0 \leq v \leq b$

Finally, $\int_{Side} \mathbf{F} \cdot dS$

$g(u, v) = (a\cos u, a\sin u, v), 0 \le u \le 2\pi, 0 \le v \le b$

$g_u = (-a\sin u, a\cos u, 0), \;\; g_v = (0, 0, 1)$

$$g_u \times g_v = \text{ det } \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin u & a\cos u & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

=(expanding along bottom row) $(a\cos u, a\sin u, 0)$

Thus $|g_u \times g_v| = \sqrt{a^2\cos^2 u + a^2\sin^2 u + 0^2} = a$

Also $F(g(u, v)) = (a\cos u, a\sin u, v)$ so $F(g(u, v)) \cdot (g_u \times g_v) = a^2\cos^2 u + a^2\sin^2 u + 0 = a^2$.

so $\int_{Side} \mathbf{F} \cdot dS = \int_{v=0}^{b} \int_{u=0}^{2\pi} a^2 \, du \, dv = 2\pi a^2 b$

Putting it altogether: $\int_S \mathbf{F} \cdot dS$

$= \int_{Bottom} \mathbf{F} \cdot dS + \int_{Top} \mathbf{F} \cdot dS + \int_{Side} \mathbf{F} \cdot dS = 0 + \pi a^2 b + 2\pi a^2 b = 3\pi a^2 b$

On The Other Hand, we compute $\int_R$ div $\mathbf{F}$

$\mathbf{F} = (x, y, z)$

div $\mathbf{F} = 1 + 1 + 1 = 3$

The solid R is more easily described in polar coordinates

$0 \le \theta \le 2\pi \quad 0 \le r \le a \quad 0 \le z \le b$.

$$\int_R \text{div } \mathbf{F} = \int_{\theta=0}^{2\pi} \int_{z=0}^{b} \int_{r=0}^{a} \text{div } \mathbf{F} r dr dz d\theta = \int_{\theta=0}^{2\pi} \int_{z=0}^{b} \int_{r=0}^{a} 3r dr dz d\theta$$

$$\int_{\theta=0}^{2\pi} \int_{z=0}^{b} 3\frac{r^2}{2}|_{r=0}^{a} dz d\theta = \int_{\theta=0}^{2\pi} \int_{z=0}^{b} \frac{3}{2} a^2 dz d\theta = \int_{\theta=0}^{2\pi} \frac{3}{2} a^2 b d\theta = 2\pi \frac{3}{2} a^2 b$$

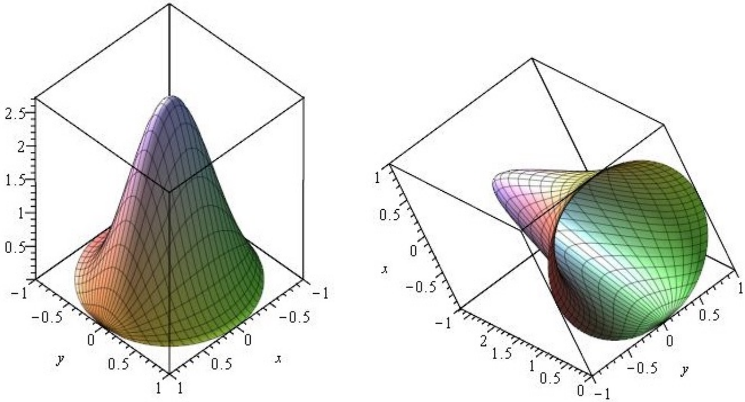
$$= 3a^2 b\pi$$

Example: $\mathbf{F} = (e^y \cos z, \sqrt{x^3+1}\sin z, x^2+y^2+3)$

div $\mathbf{F} = 0 + 0 + 0 = 0$

so $\int_R$ div $\mathbf{F} = 0$ for any region in $\mathbb{R}^3$.

Let S be graph of $z = (1 - x^2 - y^2)e^{1-x^2-3y^2})$ for $z \geq 0$ oriented by outward pointing unit normal vector.



Finding $\int_S \mathbf{F} \cdot d\sigma$ directly is impossible.

A Clever Way To Find $\int_S \mathbf{F} \cdot d\sigma$ indirectly.

Cap the Surface with a Disk so New Surface Bounds a 3-Dimensional Region

Form closed surface $S \cup S'$ where S' is the disk of radius 1 $(x^2 + y^2 = 1)$ in $z = 0$ plane.

Then $\int_{\partial r} \mathbf{F} = \int_{S \cup S'} \mathbf{F} = \int_S \mathbf{F} + \int_{S'} \mathbf{F}$
But by Gauss's Theorem, this integral equals 0.
Hence $\int_S \mathbf{F} = -\int_{S'} \mathbf{F}$

Now

$$\int_{S'} \mathbf{F} = -\int (--, --, x^2 + y^2 + 3) \cdot (0, 0, -1) = \int x^2 + y^2 + 3 \, dx \, dy$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} (r^2 + 3) \, r \, dt \, d\theta = \frac{7}{2}\pi$$

Next Time:

Stokes's Theorem

$$\int_S \text{curl } \mathbf{F} = \int_{\partial S} \mathbf{F}$$

S is a Surface in $\mathbb{R}^3$