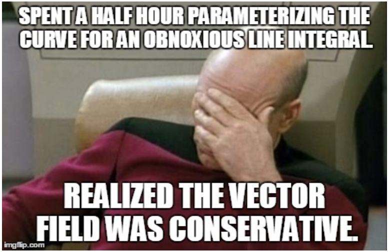
MATH 223: Multivariable Calculus



Class 33: December 5, 2022

Handouts

Notes on Assignment 30
Assignment 31
Conservative Vector Fields
Surface Integrals

Announcements

Independent Projects Due Friday

Today

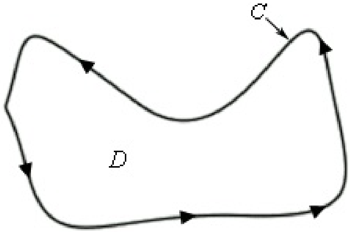
Proof of Green's Theorem

Conservative Vector Fields

Surface Integrals

Green's Theorem in the Plane

$$\iint_D \text{curl}\,\mathbf{F} = \int_\gamma \mathbf{F}$$



D is bounded plane region.

$C = \gamma$ is piecewise smooth boundary of D

F and G are continuously differentiable functions defined on D

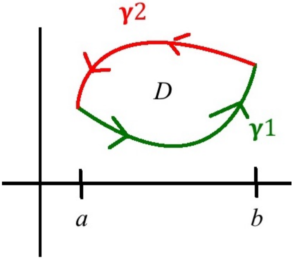
Then

$$\int\int (G_x - F_y)dxdy = \int_\gamma (F, G)$$

where γ is parametrized so it is traced once with D on the left.

Proof of Green's Theorem in an Elementary Case

Case : Boundary of D is made up of the graphs of two functions defined on interval $[a, b]$.



Ingredients:

Vector Field **F** $= (F, G) = (F, 0) + (0, G)$

$\gamma_1 =$ image of g_1

$\gamma_2 =$ image of g_2

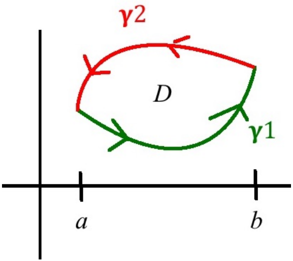
Need to show $\iint_D [G_x - Fy] = \int_\gamma \mathbf{F} = \int_\gamma [(F, 0) + (0, G)]$

Will show $\iint_D -Fy = \int_\gamma (F, 0)$

Need to show $\iint_D [G_x - Fy] = \int_\gamma \mathbf{F} = \int_\gamma [(F, 0) + (0, G)]$

Will show $\iint_D -Fy = \int_\gamma (F, 0)$

We tackle the line integral first. Start with γ_1



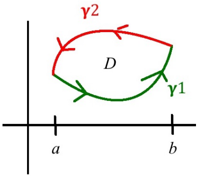
We can parametrize γ_1 by a function $g(t) = (t, \phi(t))$ for $a \leq t \leq b$

Then $g'(t) = (1, \phi_1'(t))$

Now $(F, 0) \cdot g'(t) = (F, 0) \cdot (1, \phi_1'(t)) = F = F(t, \phi_1(t))$

so $\int_{\gamma_1} (F, 0) = \int_a^b F(t, \phi_1(t))\, dt$

Now we take up γ_2



Consider Parametrization of γ_2 as $g(t) = (t, \phi_2(t)), a \leq t \leq b$.
This would actually traces out γ_2 in the opposite direction. It is the parametrization of $-\gamma_2$

Again we have $g'(t) = (1, \phi_2^{'})$ and $(F, 0) \cdot g'(t) = F(t, \phi_2(t))$

so $\int_{-\gamma_2} (F, 0) = \int_a^b F(t, \phi_2(t))$.

Thus $\int_{-\gamma_2} (F, 0) = -\int_{\gamma_2} = -\int_a^b F(t, \phi_2(t))$.

Finally, $\int_{\gamma} (F, 0) = \int_{\gamma_1} (F, 0) + \int_{\gamma_2} (F, 0)$

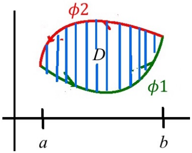
$= \int_a^b F(t, \phi_1(t))\, dt - \int_a^b F(t, \phi_2(t))\, dt$

$$\boxed{\int_{\gamma} (F, 0) = \int_a^b F(t, \phi_1(t)) - F(t, \phi_2(t))\, dt}$$

Goal: Show $\iint_D -Fy = \int_\gamma (F,0)$

So far: $\int_\gamma (F,0) = \int_a^b F(t,\phi_1(t)) - F(t,\phi_2(t))\,dt$

Now turn to the curl part:



$$\iint_D -F_y = -\iint_D F_y = \int_{x=a}^{x=b} \int_{y=\phi_1(x)}^{y=\phi_2(x)} -Fy(x,y)\,dy\,dx$$

$$= -\int_a^b \left[F(x,\phi_2(x)) - F(x,\phi_1(x)\right]\,dx$$

$$= -\int_a^b \left[F(t,\phi_2(t)) - F(t,\phi_1(t)\right]\,dt(\text{ let } t = x)$$

$$= \int_a^b \left[F(t,\phi_1(t)) - F(t,\phi_2(t)\right]\,dt$$

Conservative Vector Fields

F is continuously differentiable vector field in the plane
F : $\mathbb{R}^2 \to \mathbb{R}^2$ with **F**$(x, y) = (F(x, y), G(x, y))$ where F and G are each real-valued functions.

Here curl **F** is a real-valued function $G_x - F_y$

Green's Theorem: $\int_D$ curl **F** $= \int_\gamma$ **F**

Three Important Properties of Vector Fields

A **F** is **CONSERVATIVE** means **F** $= \nabla f$ for some $f : \mathbb{R}^2 \to \mathbb{R}^1$

B **F** is **IRROTATIONAL** means curl **F** $= 0$

C **F** is **PATH INDEPENDENT** means $\int_{\gamma_1}$ **F** $= \int_{\gamma_2}$ **F** for any paths γ_1 and γ_2 from **a** to **b** where **a** and **b** are any points in the plane.

Major Goal: Show THESE PROPERTIES ARE EQUIVALENT

A implies **B**

A **F** is **CONSERVATIVE** means **F** $= \nabla f$ for some $f : \mathbb{R}^2 \to \mathbb{R}^1$

B **F** is **IRROTATIONAL** means curl **F** $= 0$

Suppose **F** is Conservative

Then $(F, G) =$ **F** $= \nabla f = (f_x, f_y)$ so $f_x = F$ and $f_y = G$

Thus $G_x = f_{yx}$ and $F_y = f_{xy}$

so curl **F** $= G_x - F_y = f_{yx} - f_{xy} = 0$

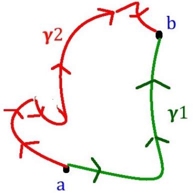
by equality of mixed partials.

B implies **C** will follow from Green's Theorem

B **F** is **IRROTATIONAL** means curl $\mathbf{F} = 0$

C **F** is **PATH INDEPENDENT** means $\int_{\gamma_1} \mathbf{F} = \int_{\gamma_2} \mathbf{F}$ for any paths γ_1 and γ_2 from **a** to **b** where **a** and **b** are any points in the plane.

Let **a** and **b** are any points in the plane and γ_1 and γ_2 two paths from **a** to **b**. Then $-\gamma_1$ runs from **b** to **a**



and $\gamma = \gamma_1 - \gamma_2$ is a loop that begins and ends at **a**

Let D be the enclosed region.

By Green's Theorem $\int_\gamma \mathbf{F} = \iint_D$ curl $\mathbf{F} = \iint_D 0 = 0$

Thus $0 = \int_\gamma \mathbf{F} = \int_{\gamma_1 - \gamma_2} \mathbf{F} = \int_{\gamma_1} \mathbf{F} - \int_{\gamma_2} \mathbf{F}$

Hence $\int_{\gamma_2} \mathbf{F} = \int_{\gamma_1} \mathbf{F}$

C implies **A**

C **F** is **PATH INDEPENDENT** means $\int_{\gamma_1} \mathbf{F} = \int_{\gamma_2} \mathbf{F}$ for any paths γ_1 and γ_2 from **a** to **b** where **a** and **b** are any points in the plane.

A **F** is **CONSERVATIVE**means $\mathbf{F} = \nabla f$ for some $f : \mathbb{R}^2 \to \mathbb{R}^1$

Idea:
Fix $\mathbf{x}_0$ in $\mathbb{R}^n$ and let $\mathbf{x}$ be arbitrary point in $\mathbb{R}^n$.
Let γ be a curve from $\mathbf{x}_0$ to $\mathbf{x}$.
Then $\int_\gamma \mathbf{F}$ will be a function of $\mathbf{x}$ whose gradient is $\mathbf{F}$.

Theorem Let **F** be a continuous vector field defined in a polygonally connected open set D of $\mathbb{R}^n$. If the line integral $\int_\gamma \mathbf{F}$ is independent of piecewise smooth path γ from $\mathbf{x}_0$ to $\mathbf{x}$ in D, then if $f(\mathbf{x}) = \int_\gamma \mathbf{F}$, it is true that $\nabla f = \mathbf{F}$.

Example $\mathbf{F}(x, y) = (3x^2 + y, e^y + x)$

Here $\mathbf{F} = (F, G)$ so $F(x, y) = 3x^2 + y, G(x, y) = e^y + x$

Hence $F_y = 1, G_x = 1$ so curl $\mathbf{F} = G_x - F_y = 0$

Let's build f so its gradient $\nabla f = (f_x, f_y) = (3x^2 + y, e^y + x)$

We need $f_x = 3x^2 + y$ so do "partial integration with respect to x":

$f(x) = x^3 + yx + g(y)$. [Why is there $g(y)$?]

Then $f_y = 0 + x + g'(y)$ which should equal $x + e^y$ so need

$$g'(y) = e^y$$

which we can get by letting $g(y) = e^y$.

Hence we can choose $f(x, y) = x^3 + yx + e^y + C$.

Let's build the potential function in a different way using the theorem with $\mathbf{F}(x,y) = (3x^2 + y, e^y + x)$

Pick $\mathbf{x}_0 = (0,0)$ and let $\mathbf{x} = (x,y)$ be an arbitrary point. Choose the straight line between them as the path γ with parametrization

$g(t) = (xt, yt), 0 \leq t \leq 1$ so $g'(t) = (x,y)$

Then $\mathbf{F}(g(t)) = F(xt, yt) = (3x^2t^2 + yt, e^{yt} + x)t$

so $\mathbf{F}(g(t)) \cdot g'(t) = (3x^2t^2 + yt, e^{yt} + xt) \cdot (x,y)$

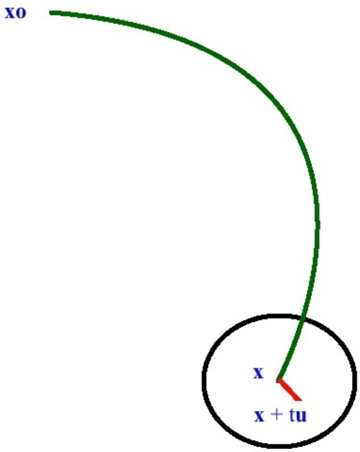
$= 3x^3t^2 + xyt + ye^{yt} + xyt = 3x^3t^2 + 2xyt + ye^{yt}$

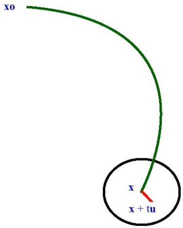
Now $\int_\gamma \mathbf{F} = \int_0^1 (3x^3t^2 + 2xyt + ye^{yt})\, dt$

$= \left[x^3t^3 + xyt^2 + e^{yt}\right]_{t=0}^{t=1}$

$= (x^3 + xy + e^y) - (0 + 0 + 1) = x^3 + xy + e^y - 1$

Theorem Let $\mathbf{F}$ be a continuous vector field defined in a polygonally connected open set D of $\mathbb{R}^n$. If the line integral $\int_\gamma \mathbf{F}$ is independent of piecewise smooth path γ from $\mathbf{x}_0$ to $\mathbf{x}$ in D, then if $f(\mathbf{x}) = \int_\gamma \mathbf{F}$, it is true that $\nabla f = \mathbf{F}$.





Let g be parametrization of line segment from $\mathbf{x}$ to $\mathbf{x} + t\mathbf{u}$ so
$g(t) = \mathbf{x} + v\mathbf{u}, 0 \leq v \leq t$ and $g'(t) = \mathbf{u}$

$$f(\mathbf{x} + t\mathbf{u}) - f(\mathbf{x}) = \int_{\mathbf{x}_0}^{\mathbf{x}+t\mathbf{u}} \mathbf{F} - \int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{F} = \int_{\mathbf{x}}^{\mathbf{x}+t\mathbf{u}} \mathbf{F}(\mathbf{x} + v\mathbf{u})$$
$$= \int_0^t \mathbf{F}(\mathbf{x} + v\mathbf{u}) \cdot \mathbf{u}\, dv$$

To find $\frac{\partial f}{\partial x_j}(\mathbf{x})$, let $\mathbf{u}$ be the unit vector $\mathbf{e}_j = (0, 0, .. . , 1, 0, 0. . . 0)$ in the jth direction.

$$\frac{\partial f}{\partial x_j}(\mathbf{x}) = \lim_{t \to 0} \frac{f(\mathbf{x} + t\mathbf{u}) - f(\mathbf{x})}{t}$$

$$= \lim_{t \to 0} \frac{1}{t} \int_0^t \mathbf{F}(\mathbf{x} + v\mathbf{u}) \cdot \mathbf{u} \, dv$$

$$= \lim_{t \to 0} \frac{1}{t} \int_0^t \mathbf{F}(\mathbf{x} + v\mathbf{e}_j) \cdot \mathbf{e}_j \, dv$$

But this last expression is the derivative of the integral with respect to t evaluated at $t = 0$ which is $\mathbf{F} \cdot \mathbf{e}_j = F_j(\mathbf{x})$ (Using Fundamental Theorem of Calculus)

Symmetry of Jacobian Matrix for Conservative Vector Field

Let **F** $= (F(x,y), G(x,y))$ be a conservative vector field in the plane which we can recognized by $G_x = F_y$

F' $= \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix}$ Note symmetry of Jacobian Matrix.

How do things generalize to higher dimensions?

Example: **F**: $\mathbb{R}^3 \to \mathbb{R}^3$ by

$$F(x,y,z) = (yz^2 + \sin y + 3x^2, xz^2 + x\cos y + e^z, 2xyz + ye^z + \frac{1}{z})$$

$$\mathbf{F'} = \begin{pmatrix} 6x & z^2 + \cos y & 2yz \\ z^2 + \cos y & -x \sin y & 2xz + e^z \\ 2yz & 2xz + e^z & 2xy + ye^z - \frac{1}{z^2} \end{pmatrix}$$

To find f so that $\nabla f = \mathbf{F}$:

Step 1: integrate first component of **F** with respect to x:

$$f(x,y,z) = yz^2x + x\sin y + x^3 + G(y,z)$$

Step 2: Take derivative of trial f respect to y and set equal to second component of **F** :

$$f_y = z^2x + x\cos y + 0 + G_y(x,y) \text{ must } = xz^2 + x\cos y + e^z$$

Need $G_y(x,y) = e^z$ so choose $G(x,y) = e^z y + H(z)$

So far, $f(x,y,z) = yz^2x + x\sin y + x^3 + e^z y + H(z)$

Step 3:Take derivative of trial f respect to z and set equal to third component of **F** ;

$$f_z(x,y,z) = 2xyz + 0 + 0 + e^z y + H'(z) \text{ must } = 2xyz + e^z y + \frac{1}{z}$$

Need $H'(z) = \frac{1}{z}$ so choose $H(x) = \ln|z| + C$

Thus

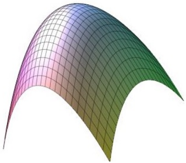
$$f(x,y,z) = f(x,y,z) = yz^2x + x\sin y + x^3 + e^z y + \ln|z| + C$$

Theorem If **F** is a conservative vector field on $\mathbb{R}^N = n$ and is continuously differentiable, then the Jacobian matrix is symmetric.

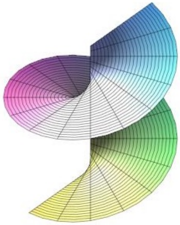
Proof: Equality of mixed partials.

Theorem Suppose **F** is a continuously differentiable vector field on $\mathbb{R}^n$ whose Jacobian matrix is symmetric. Then **F** is conservative

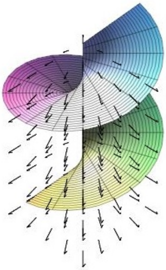
Integrating Vector Fields Over Surfaces



$g(u, v) = [u, v, -2u^2 - 3v^2]$



$g(u, v) = [u\cos v, u\sin v, v]$



Smooth Curve γ	Smooth Surface S
$g : I$ in $\mathbb{R}^1 \to \mathbb{R}^n$	$g : D$ in $\mathbb{R}^2 \to \mathbb{R}^3$
Length $= \int_I \vert g'(t)\vert \, dt$	Area $\sigma(S) = \iint_D \vert g_u \times g_v\vert dudv$
Mass $= \int_I \mu(g(t))\vert g'(t)\vert \, dt$ Line Integral:	Mass $= \iint_D \mu \, d\sigma$ Surface Integral
$\int_\gamma \mathbf{F} = \int_I \mathbf{F}(g(t)) \cdot g'(t) \, dt$	$\iint_S \mathbf{F} = \iint_D \mathbf{F}(g(u,v)) \cdot (g_u \times g_v)$

$$\iint_S \mathbf{F} = \iint_S \mathbf{F} \cdot dS = \iint_S \mathbf{F} \cdot \mathbf{N} d\sigma$$

$\Phi(\mathbf{F}, S) = \iint_S \mathbf{F}$ is **flux** of **F** across S.

Surface Integral

Let g be a function from an interval $[t_0, t_1]$ into $\mathbb{R}^n$ with image γ and mu density at $g(t)$.

Then Mass of Wire $= \int_{t_0}^{t_1} \mu(t)|g'(t)|\,dt$

If $\mu \equiv 1$, then mass $=$ length of curve $\int_{t_0}^{t_1} |g'(t)|\,dt$

Generalize To Surfaces

Let D be region in plane and $g : D \to \mathbb{R}^3$ with $g(u,v) = (g_1, g_2, g_3)$ where each component function g_i is continuously differentiable.

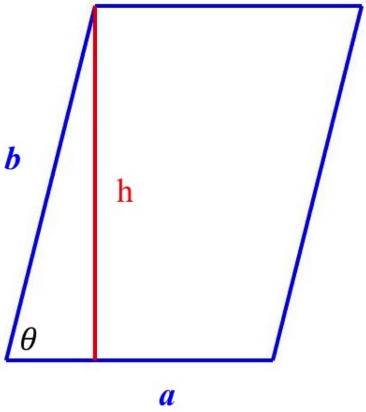
There are two natural tangent vectors: $g_u = \frac{\partial g}{\partial u}$ and $g_v = \frac{\partial g}{\partial v}$, These determine a tangent plane.

S is a **Smooth Surface** if these two vectors are linearly independent.

Note that $\frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v}$ is normal to the plane with

$$|\frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v}| = |\frac{\partial g}{\partial u}||\frac{\partial g}{\partial v}|\sin\theta$$

$=$ Area of Parallelogram Spanned by the Vectors



$$\sin\theta = \frac{h}{|\mathbf{b}|} \text{ so } h = |\mathbf{b}|\sin\theta$$

$$\text{Area of Parallelogram} = (\text{Base})(\text{Height}) = |\mathbf{a}||\mathbf{b}|\sin\theta$$

$$\mathbf{a} = g_u, \mathbf{b} = g_v$$

$$|g_u \times g_v| = |g_u||g_v|\sin\theta$$

Surface Area

$$\sigma(S) = \iint_D |\frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v}|\, dudv = \iint_D |g_u \times g_v|\, dudv$$

If $\mu(g(u,v))$ is density, then mass $=$

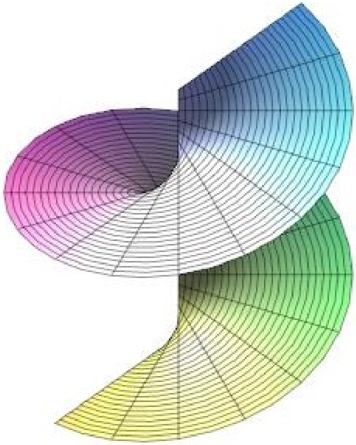
$$\iint_D \mu\, d\sigma = \iint_D \mu(g(u,v))|g_u \times g_v|\, dudv$$

Plotting Parametrized Surface in $Maple$:

$$plot3d([g1(u,v), g2(u,v), g3(u,v)], u = ..., v = ...)$$

Area of a Spiral Ramp

$$g(u, v) = (u \cos v, u \sin v, v), 0 \leq u \leq 1, 0 \leq v \leq 3\pi$$



Area of a Spiral Ramp

$$g(u,v) = (u\cos v, u\sin v, v), 0 \le u \le 1, 0 \le v \le 3\pi$$

$$g_u = (\cos v, \sin v, 0), g_v = (-u\sin v, u\cos v, 1)$$

$$g_u \times g_v = \text{ det } \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 1 \end{vmatrix}$$

$$= \left(\begin{vmatrix} \sin v & 0 \\ u\cos v & 1 \end{vmatrix}, - \begin{vmatrix} \cos v & 0 \\ -u\sin v & 1 \end{vmatrix}, \begin{vmatrix} \cos v & \sin v \\ -u\sin v & u\cos v \end{vmatrix} \right)$$

$$= (\sin v, -\cos v, u)$$

Then $|g_u \times g_v| = \sqrt{\sin^2 v + \cos^2 v + u^2} = \sqrt{1+u^2}$

Area $= \int_{v=0}^{v=3\pi} \int_{u=0}^{1} \sqrt{1+u^2}\, du\, dv$

If density is $\mu(\mathbf{x}) = u$, then

Mass $=$

$$\int_{v=0}^{v=3\pi} \int_{u=0}^{u=1} u(1+u^2)^{1/2}\, du\, dv = \int_{v=0}^{v=3\pi} \left[\tfrac{1}{3}(1+u^2)^{3/2}\right]_0^1\, dv$$

$$= \int_{v=0}^{v=3\pi} \tfrac{1}{3}[2^{3/2} - 1^{3/2}]\, dv = 3\pi\tfrac{1}{3}[2^{3/2} - 1] = \pi[2^{3/2} - 1]$$