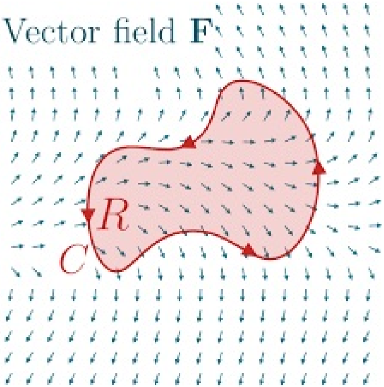
MATH 223: Multivariable Calculus



Class 32: December 2, 2022

Notes on Assignment 29
Assignment 30
Green'sTheorem
Notes on Exam 3

Announcements

Today

More Green's Theorem

Conservative Vector Fields

Divergence of a Vector Field

Definition div **F** $=$ trace of **F'**, the Jacobi Matrix

In general, div **F** is a real -valued function of n variables.

Curl of a Vector Field

Curl measures local tendency of a vector field and its flow lines to circulate around some axis.

The curl of a vector field is itself a vector field.

Setting; $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is our vector field

$\mathbf{F} = (F_1, F_2, F_3)$ so $\mathbf{F}(x, yz) = (F_1(x,y,z), F_2(x,y,z), F_3(x,y,z))$

Formal Definition: curl $\mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$

Mnemonic Device:

$$\text{curl } \mathbf{F} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}. \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{pmatrix}$$

Expand along first row:

$$\text{curl } \mathbf{F} = \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_2 & F_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ F_1 & F_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ F_1 & F_2 \end{vmatrix} \mathbf{k}$$

Scalar Curl for Vector Fields in Plane

F $= (F, G, 0)$ where$F(x, y)$ and $G(x, y)$are functions only of x and y.

Then curl **F** $= (0, 0, G_x - F_y)$

Note: Curl and Conservative Vector Field

Suppose **F** $= (F, G, 0)$ is gradient field with **F** $= \nabla f$.

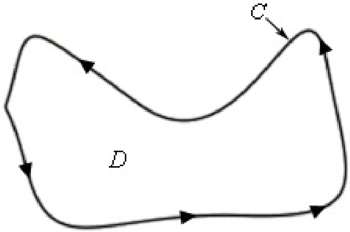
Then $F = f_x$ and $G = f_y$

In this case, Curl **F** $= (0, 0, f_{yx} - f_{xy}) = (0, 0, 0)$

by Clairault's Theorem on Equality of Mixed Partials.

Green's Theorem in the Plane

$$\iint_D \text{curl}\, \mathbf{F} = \int_\gamma \mathbf{F}$$



D is bounded plane region.

$C = \gamma$ is piecewise smooth boundary of D

F and G are continuously differentiable functions defined on D

Then

$$\int\int (G_x - F_y) dx dy = \int_\gamma (F, G)$$

where γ is parametrized so it is traced once with D on the left.

Using Green's Theorem

(1) Compute $\iint_D \text{curl}\, \mathbf{F}$ by using $\int_\gamma \mathbf{F}$

(2) Compute $\int_\gamma \mathbf{F}$ by using $\iint_D \text{curl}\, \mathbf{F}$

Using Green's Theorem

Compute $\int_\gamma \mathbf{F}$ by using $\iint_D$ curl $\mathbf{F}$

<u>Example</u> Let $\mathbf{F}(x, y) = (\frac{1}{y}\cos\frac{x}{y}, -\frac{x}{y^2}\cos\frac{x}{y})$

Compute $\int_\gamma \mathbf{F}$ as $\iint_D (G_x - F_y)$

Here $G_x = (-\frac{x}{y^2})_x \cos\frac{x}{y} + -\frac{x}{y^2}(\cos\frac{x}{y})_x$

$= -\frac{1}{y^2}\cos\frac{x}{y} - \frac{x}{y^2}(-\sin\frac{x}{y})(\frac{1}{y})$

$= -\frac{1}{y^2}\cos\frac{x}{y} + \frac{x}{y^3}(\sin\frac{x}{y})$

Similarly, $F_y = -\frac{1}{y^2}\cos\frac{x}{y} + \frac{1}{y}(-\sin\frac{x}{y})(\frac{-x}{y^2})$

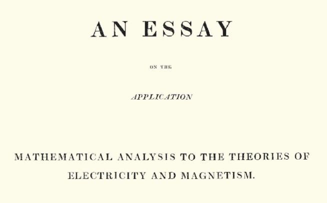
$= -\frac{1}{y^2}\cos\frac{x}{y} + \frac{x}{y^3}(+\sin\frac{x}{y})$

So $G_x - F_y = 0$.

Hence $\int_\gamma \mathbf{F} = 0$

George Green
1793 – 1841

Mikhail Ostrogradsky
1801 – 1861



AN ESSAY

ON THE

APPLICATION

MATHEMATICAL ANALYSIS TO THE THEORIES OF
ELECTRICITY AND MAGNETISM.

Gauss' Theorem

$$\text{Green: } \iint_D \text{curl } \mathbf{F} = \int_\gamma \mathbf{F}$$

If $\mathbf{F} = (F_1, F_2)$ then curl $\mathbf{F} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$

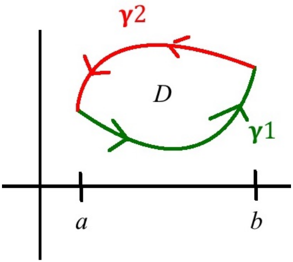
Apply Green's Theorem to $\mathbf{H} = (-G, F)$ where $\mathbf{F} = (F, G)$
$\int_\gamma \mathbf{H} = \iint_D$ curl $(F_x - (-G_y)) = \iint_D (F_x + G_y) = \iint_D$ div $\mathbf{F}$

On the other hand, $\int_\gamma \mathbf{H} = \int_a^b \mathbf{H} \cdot \mathbf{g'} = \int_a^b (-G, F) \cdot (g_1^{'}, g_2^{'})$
$\int_a^b (-G, F) \cdot (g_1^{'}, g_2^{'}) = \int_a^b -G g_1^{'} + F g_2^{'} = \int_a^b (F, G) \cdot (g_2^{'}, -g_1^{'})$
Observe $(g_2^{'}, -g_1^{'}) \cdot (g_1^{'}, g_2^{'}) = g_1^{'} g_2^{'} - g_1^{'} g_2^{'} = 0$
So $(g_2^{'}, -g_1^{'})$ is orthogonal to the tangent vector so it is a normal vector $\mathbf{N}$.
Thus $\int_\gamma \mathbf{H} = \int_a^b (F, G) \cdot (g_2^{'}, -g_1^{'}) = \int_a^b (F, G) \cdot \mathbf{N} = \int_\gamma \mathbf{F} \cdot \mathbf{N}$

Putting everything together: $\boxed{\iint_D \text{div } \mathbf{F} = \int_\gamma \mathbf{H} = \int_\gamma \mathbf{F} \cdot \mathbf{N}}$

Proof of Green's Theorem in an Elementary Case

Case : Boundary of D is made up of the graphs of two functions defined on interval $[a, b]$.



Ingredients:

Vector Field $\mathbf{F} = (F, G) = (F, 0) + (0, G)$

$\gamma_1 =$ image of g_1

$\gamma_2 =$ image of g_2

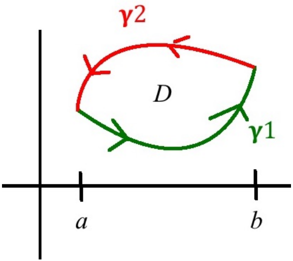
Need to show $\iint_D [G_x - Fy] = \int_\gamma \mathbf{F} = \int_\gamma [(F, 0) + (0, G)]$

Will show $\iint_D -Fy = \int_\gamma (F, 0)$

Need to show $\iint_D [G_x - Fy] = \int_\gamma \mathbf{F} = \int_\gamma [(F, 0) + (0, G)]$

Will show $\iint_D -Fy = \int_\gamma (F, 0)$

We tackle the line integral first. Start with γ_1



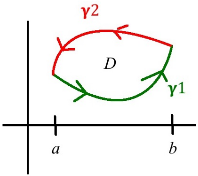
We can parametrize γ_1 by a function $g(t) = (t, \phi(t))$ for $a \leq t \leq b$

Then $g'(t) = (1, \phi_1'(t))$

Now $(F, 0) \cdot g'(t) = (F, 0) \cdot (1, \phi_1'(t)) = F = F(t, \phi_1(t))$

so $\int_{\gamma_1} (F, 0) = \int_a^b F(t, \phi_1(t))\, dt$

Now we take up γ_2



Consider Parametrization of γ_2 as $g(t) = (t, \phi_2(t)), a \leq t \leq b$.
This would actually traces out γ_2 in the opposite direction. It is the parametrization of $-\gamma_2$

Again we have $g'(t) = (1, \phi_2^{'})$ and $(F, 0) \cdot g'(t) = F(t, \phi_2(t))$

so $\int_{-\gamma_2} (F, 0) = \int_a^b F(t, \phi_2(t))$.

Thus $\int_{-\gamma_2} (F, 0) = -\int_{\gamma_2} = -\int_a^b F(t, \phi_2(t))$.

Finally, $\int_{\gamma} (F, 0) = \int_{\gamma_1} (F, 0) + \int_{\gamma_2} (F, 0)$

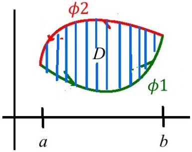
$= \int_a^b F(t, \phi_1(t))\, dt - \int_a^b F(t, \phi_2(t))\, dt$

$$\boxed{\int_{\gamma} (F, 0) = \int_a^b F(t, \phi_1(t)) - F(t, \phi_2(t))\, dt}$$

Goal: Show $\iint_D -Fy = \int_\gamma (F,0)$

So far: $\int_\gamma (F,0) = \int_a^b F(t,\phi_1(t)) - F(t,\phi_2(t))\,dt$

Now turn to the curl part:



$$\iint_D -Fy = -\iint_D F_y = \int_{x=a}^{x=b} \int_{y=\phi_1(x)}^{y=\phi_2(x)} -Fy(x,y)\,dy\,dx$$

$$= -\int_a^b \left[F(x,\phi_2(x)) - F(x,\phi_1(x)\right]\,dx$$

$$= -\int_a^b \left[F(t,\phi_2(t)) - F(t,\phi_1(t)\right]\,dt(\text{ let } t = x)$$

$$= \int_a^b \left[F(t,\phi_1(t)) - F(t,\phi_2(t)\right]\,dt$$

Conservative Vector Fields

F is continuously differentiable vector field in the plane

F : $\mathbb{R}^2 \to \mathbb{R}^2$ with $\mathbf{F}(x, y) = (F(x, y), G(x, y))$ where F and G are each real-valued functions.

Here curl **F** is a real-valued function $G_x - F_y$

Green's Theorem: $\int_D \text{curl } \mathbf{F} = \int_\gamma \mathbf{F}$

Three Important Properties of Vector Fields

A: **F** is **CONSERVATIVE** means $\mathbf{F} = \nabla f$ for some $f : \mathbb{R}^2 \to \mathbb{R}^1$

B: **F** is **IRROTATIONAL** means curl $\mathbf{F} = 0$

C: **F** is **PATH INDEPENDENT** means $\int_{\gamma_1} \mathbf{F} = \int_{\gamma_2} \mathbf{F}$ for any paths γ_1 and γ_2 from **a** to **b** where **a** and **b** are any points in the plane.

Major Goal: Show THESE PROPERTIES ARE EQUIVALENT

A implies **B**

A **F** is **CONSERVATIVE**means $\mathbf{F} = \nabla f$ for some $f : \mathbb{R}^2 \to \mathbb{R}^1$

B **F** is **IRROTATIONAL** means curl $\mathbf{F} = 0$

Suppose **F** is Conservative

Then $(F, G) = \mathbf{F} = \nabla f = (f_x, f_y)$ so $f_x = F$ and $f_y = G$

Then $G_x = f_{yx}$ and $F_y = f_{xy}$

so curl $\mathbf{F} = G_x - F_y = f_{yx} - f_{xy} = 0$

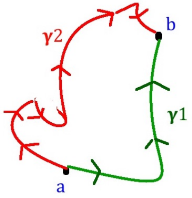
by equality of mixed partials.

B implies **C** will follow from Green's Theorem

B **F** is **IRROTATIONAL** means curl **F** $= 0$

C **F** is **PATH INDEPENDENT** means $\int_{\gamma_1} \mathbf{F} = \int_{\gamma_2} \mathbf{F}$ for any paths γ_1 and γ_2 from **a** to **b** where **a** and **b** are any points in the plane.

Let **a** and **b** are any points in the plane and γ_1 and γ_2 two paths from **a** to **b**. Then $-\gamma_1$ runs from **b** to **a**



and $\gamma = \gamma_1 - \gamma_2$ is a loop that begins and ends at **a**

Let D be= the enclosed region.

By Green's Theorem $\int_\gamma \mathbf{F} = \iint_D$ curl $\mathbf{F} = \iint_D 0 = 0$

Thus $0 = \int_\gamma \mathbf{F} = \int_{\gamma_1 - \gamma_2} \mathbf{F} = \int_{\gamma_1} \mathbf{F} - \int_{\gamma_2} \mathbf{F}$

Hence $\int_{\gamma_2} \mathbf{F} = \int_{\gamma_1} \mathbf{F}$

C implies **A**

C **F** is **PATH INDEPENDENT** means $\int_{\gamma_1} \mathbf{F} = \int_{\gamma_2} \mathbf{F}$ for any paths γ_1 and γ_2 from **a** to **b** where **a** and **b** are any points in the plane.

A **F** is **CONSERVATIVE** means $\mathbf{F} = \nabla f$ for some $f : \mathbb{R}^2 \to \mathbb{R}^1$