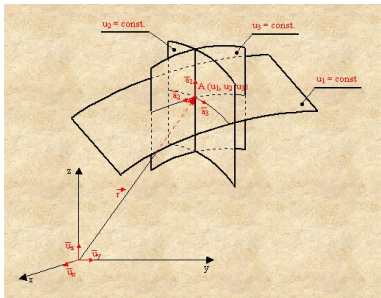


MATH 223: Multivariable Calculus



Class 21: October 31, 2022

Happy
Halloween





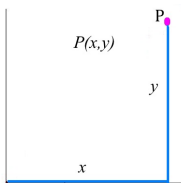
Notes on Assignments 18 and 19
Assignments 20 and 21
Curvilinear Coordinates

Announcements

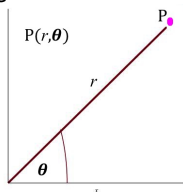
Exam 2: Wednesday
7 PM –?
101 Warner

Review Basic Theorems About Integration from Calculus I

Today:
Curvilinear Coordinates
Coordinate Systems in Plane and Space
Plane



Cartesian



Polar

Newton(1671) [Not published until after death]

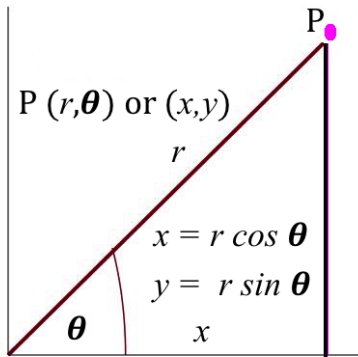
Jacob Bernoulli (1691)

In Polar Coordinates, Circles and Lines Through Origin Have
Simple Equations:

Circle: $r = 4$

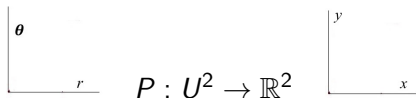
Line: $\theta = \pi/6$

Relationship Between Polar and Cartesian

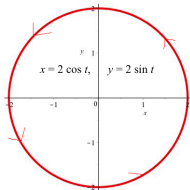
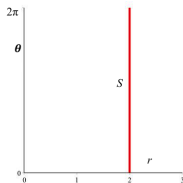


Linear Algebra Perspective

$$P \begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} r \cos \theta \\ r \sin \theta \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \text{ for } 0 < r < \infty \\ 0 \leq \theta \leq 2\pi$$

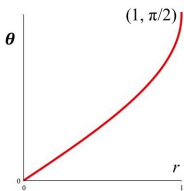


Example: P sends this line segment S to circle of radius 2 with center at origin.



$$[r, \theta] = [2, t], 0 \leq t \leq 2\pi$$

Example: $\theta = t, r = \sin t = \sin \theta$



	$r = \sin t$	$\theta = t$
0	0	0
$\pi/6$	$1/2$	$\pi/6$
$\pi/4$	$\sqrt{2}/2$	$\pi/4$
$\pi/2$	1	$\pi/2$

Then $x = r \cos \theta = \sin \theta \cos \theta$ and $y = r \sin \theta = \sin \theta \sin \theta$

So $x^2 = \sin^2 \theta \cos^2 \theta, y^2 = \sin^2 \theta \sin^2 \theta$

and then $x^2 + y^2 = \sin^2 \theta (\cos^2 \theta + \sin^2 \theta) = \sin^2 \theta \times 1 = \sin^2 \theta = y$

Thus $x^2 + y^2 - y = 0$.

Complete the square in y :

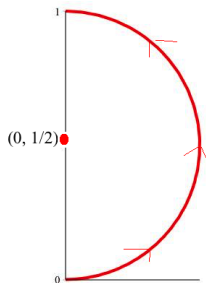
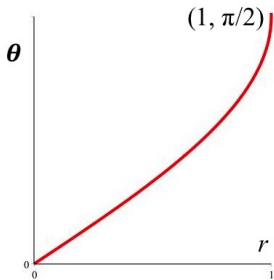
$$x^2 + y^2 - y + \frac{1}{4} = \frac{1}{4} \implies x^2 + (y - \frac{1}{2})^2 = \frac{1}{4}$$

which is the equation of a circle with center at $(0, 1/2)$ and radius $1/2$.

$$x^2 + y^2 - y + \frac{1}{4} = \frac{1}{4} \implies x^2 + (y - \frac{1}{2})^2 = \frac{1}{4}$$

which is the equation of a circle
with center at $(0, 1/2)$ and radius $1/2$.

The image is the right half of the circle:



Think of P as a function from \mathbb{R}^2 to \mathbb{R}^2 . Then

$$P' = \begin{pmatrix} \frac{\partial}{\partial r}(r \cos \theta) & \frac{\partial}{\partial \theta}(r \cos \theta) \\ \frac{\partial}{\partial r}(r \sin \theta) & \frac{\partial}{\partial \theta}(r \sin \theta) \end{pmatrix}$$

$$P' = \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix} \implies P'(\pi/6) = \begin{pmatrix} \sqrt{3}/2 & -1/4 \\ 1/2 & \sqrt{3}/4 \end{pmatrix}$$

Previous Example

$$g(t) = [\sin t, t]$$

$$\text{so } g'(t) = [\cos t, 1]$$

$$\text{At } t = \pi/6, g'(\pi/6) = [\sqrt{3}/2, 1]$$

$$g : \mathbb{R}^1 \rightarrow \mathbb{U}^2 \text{ and } P : \mathbb{U}^2 \rightarrow \mathbb{R}^2$$

$$(P \circ g) : \mathbb{R}^1 \rightarrow \mathbb{R}^2$$

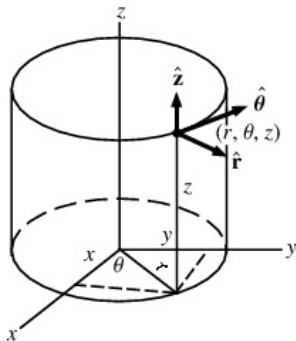
$$(P \circ g)' = P'(g) \cdot g'$$

Evaluate at $\pi/6$:

$$\begin{pmatrix} \sqrt{3}/2 & -1/4 \\ 1/2 & \sqrt{3}/4 \end{pmatrix} \begin{pmatrix} \sqrt{3}/2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ \sqrt{3}/2 \end{pmatrix}$$

Coordinate Systems in 3-Space

Cylindrical Coordinates: (r, θ, z) .



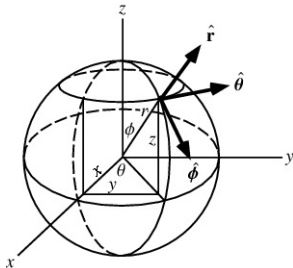
$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$z = z$$

Coordinate Systems in 3-Space

Spherical Coordinates: $(\rho, \theta, \phi) = (r, \theta, \phi)$



r = distance between origin and point

θ = project down to xy -plane

ϕ = rotation down from vertical axis

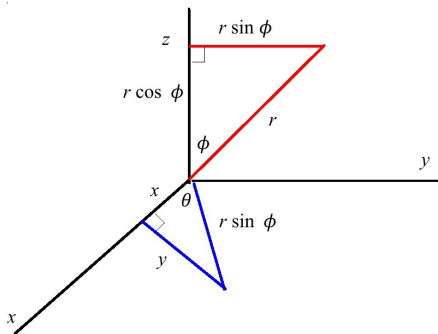
r = distance between origin and point $x = r \sin \phi \cos \theta$

θ = project down to xy -plane. $y = r \sin \phi \sin \theta$

ϕ = rotation down from vertical axis $z = r \cos \phi$

Some authors use ρ instead of r .

Converting from Spherical To Cartesian

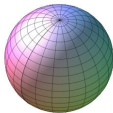


$$\cos \phi = \frac{z}{r} \implies z = r \cos \phi$$

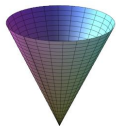
$$\cos \theta = \frac{x}{r \sin \phi} \implies x = r \sin \phi \cos \theta$$

$$\sin \theta = \frac{y}{r \sin \phi} \implies y = r \sin \phi \sin \theta$$

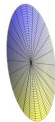
$$S : \begin{pmatrix} r \\ \phi \\ \theta \end{pmatrix} \rightarrow \begin{pmatrix} r \sin \phi \cos \theta \\ r \sin \phi \sin \theta \\ r \cos \phi \end{pmatrix}$$



$r = \text{Constant}$
Sphere



$\phi = \text{Constant}$
Cone



$\theta = \text{Constant}$
Plane

Jacobian Matrices

$$\begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix}$$

$$\begin{pmatrix} \cos \theta & -r \sin \theta & 0 \\ \sin \theta & r \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \sin \phi \cos \theta & r \cos \phi \cos \theta & -r \sin \phi \sin \theta \\ \sin \phi \sin \theta & r \cos \phi \sin \theta & r \sin \phi \cos \theta \\ \cos \phi & -r \sin \phi & 0 \end{pmatrix}$$