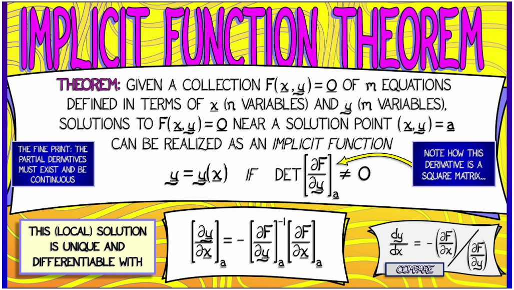
MATH 223: Multivariable Calculus



Class 17: October 21, 2022

- Notes on Assignment 15
- Assignment 16

Today

Finding a Potential Function

Implicit Differentiation II

Implicit Function Theorem

Example: Find a potential function f if

$$\nabla f(x,y) = (2x\ln(xy) + x - y^3, \frac{x^2}{y} - 3y^2x)$$

Step 1: Check Equality Of Mixed Partials

$$f_x(x,y) = 2x\ln(xy) + x - y^3 \implies f_{xy} = 2x\frac{1}{xy}x - 3y^2 = \frac{2x}{y} - 3y^2$$

$$f_y(x,y) = \frac{x^2}{y} - 3y^2x \implies f_{yx} = \frac{2x}{y} - 3y^2$$

Step 2: Integrate with respect to one of the variables

Here we will integrate f_y with respect to y so f has the form

$$f(x,y) = \int \frac{x^2}{y} - 3y^2x\,dy = x^2\ln y - y^3x + H(x)$$

for some function H of x.

Step 3: Take partial derivative of the result of Step 2 with respect to the other variable to see how close we are to the result we want. Fix the difference by adjusting the "constant" of integration.

With $f(x,y) = x^2\ln y - y^3x + H(x)$, we have

$$f_x(x,y) = 2x\ln y - y^3 + H'(x)$$

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which we want equal to

$$2x \ln(xy) + x - y^3 = 2x \ln x + 2x \ln y + x - y^3$$

Thus we need $H'(x) = 2x \ln x + x$ so we can take
$H(x) = x^2 \ln x + C$

Step 4: Put it all together to form a potential function:

$$f(x,y) = x^2 \ln y - y^3 x + H(x) = x^2 \ln y - y^3 x + x^2 \ln x + C$$

Implicit Differentiation II

The Surface $2x^3y + yx^2 + t^2 = 0$ and the Plane $x + y + t - 1 = 0$

intersect along a Curve which contains the point
$t = 1, x = -1, y = 1$

Check: Surface: $2(-1)(1) + 1(-1)^2 + 1^2 = 0$; Plane:
$-1 + 1 + 1 - 1 = 0$

Treat x and y as unknown functions of t.
Problem: Find $x'(t)$ and $y'(t)$ at $(t, x, y) = (1, -1, 1)$

Each equation defines a surface in 3-space and intersection of two surfaces is a curve.
The curve has some parametrization $\mathbf{G}$

$$\mathbf{G}(t) = \begin{pmatrix} t \\ x(t) \\ y(t) \end{pmatrix}, \mathcal{R}^1 \to \mathcal{R}^3$$

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Consider $\mathcal{R}^1 \xrightarrow{\mathbf{G}} \mathcal{R}^3 \xrightarrow{\mathbf{F}} \mathcal{R}^2$

where $\mathbf{F}(x, y, t) = \begin{pmatrix} F_1(t) \\ F_2(t) \end{pmatrix} = \begin{pmatrix} 2x^3y + yx^2 + t^2 \\ x + y + t - 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Then $\mathbf{F}(\mathbf{G}(t)) = \mathbf{0}$ for all t

Differentiate using Chain Rule:

$$[\mathbf{F}(\mathbf{G}(t))]' = \mathbf{F}'(\mathbf{G}(t))\mathbf{G}'(t) = \begin{pmatrix} F_{1t} & F_{1x} & F_{1y} \\ F_{2t} & F_{2x} & F_{yt} \end{pmatrix} \begin{pmatrix} 1 \\ x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2t & 6x^2y + 2xy & 2x^3 + x^2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Write

$$\begin{pmatrix} 2t & 6x^2y + 2xy & 2x^3 + x^2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

as

$$\begin{pmatrix} 2t \\ 1 \end{pmatrix} + \begin{pmatrix} 6x^2y + 2xy & 2x^3 + x^2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or

$$\begin{pmatrix} 6x^2y + 2xy & 2x^3 + x^2 t \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = - \begin{pmatrix} 2t \\ 1 \end{pmatrix}$$

Multiply each side by inverse of coefficient matrix

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = - \begin{pmatrix} 6x^2y + 2xy & 2x^3 + x^2 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 2t \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = -\begin{pmatrix} 6x^2y + 2xy & 2x^3 + x^2 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 2t \\ 1 \end{pmatrix}$$

Evaluate at the given point: $t = 1, x = -1, y = 1$

$$\begin{aligned} \begin{pmatrix} x' \\ y' \end{pmatrix} &= -\begin{pmatrix} 6-2 & -2+1 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ &= -\begin{pmatrix} 4 & -1 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ &= -\frac{1}{5}\begin{pmatrix} 1 & 1 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ &= -\frac{1}{5}\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -3/5 \\ -2/5 \end{pmatrix} \end{aligned}$$

More Generally

$$\left\{\begin{matrix} F_1(x,y,t)=0 \\ F_2(x,y,t)=0 \end{matrix}\right\} \text{ define } x, y \text{ implicitly as functions of } t$$

<u>Problem</u>: Find $x'(t)$ and $y'(t)$ where $\mathbf{f}(t) = \begin{pmatrix} x \\ y \end{pmatrix}$.

Set Up: $\mathcal{R}^1 \xrightarrow{\mathbf{G}} \mathcal{R}^3 \xrightarrow{\mathbf{F}} \mathcal{R}^2$ where $\mathbf{G}(t) = \begin{pmatrix} t \\ x(t) \\ y(t) \end{pmatrix}, \mathbf{F}(t,x,y) = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$

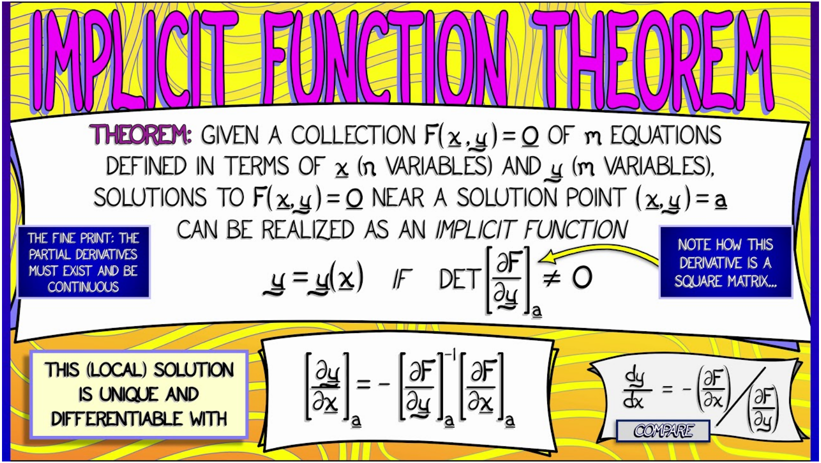
Then $\mathbf{F}(\mathbf{G}(t)) \equiv 0$ so $\mathbf{F}'(\mathbf{G}(t))\mathbf{G}'(t) = 0$ which we write as

$$(F_t, F_x, F_y)\begin{pmatrix} 1 \\ x' \\ y' \end{pmatrix} = 0 \text{ or } F_t + [F_x, F_y][\mathbf{f}'(t)] = 0$$

$$\mathbf{f}'(t) = -[F_x, F_y]^{-1} F_t$$

Here the notation is

$$F_x = \begin{pmatrix} F_{1x} \\ F_{2x} \end{pmatrix}, F_y = \begin{pmatrix} F_{1y} \\ F_{2y} \end{pmatrix}, F_t = \begin{pmatrix} F_{1t} \\ F_{2t} \end{pmatrix}$$


IMPLICIT FUNCTION THEOREM
THEOREM: GIVEN A COLLECTION $F(\underline{x},\underline{y}) = \underline{0}$ OF m EQUATIONS DEFINED IN TERMS OF $\underline{x}$ (n VARIABLES) AND $\underline{y}$ (m VARIABLES), SOLUTIONS TO $F(\underline{x},\underline{y}) = \underline{0}$ NEAR A SOLUTION POINT $(\underline{x},\underline{y}) = \underline{a}$ CAN BE REALIZED AS AN *IMPLICIT FUNCTION*
$\underline{y} = \underline{y}(\underline{x})$ IF $\det \left[\frac{\partial F}{\partial \underline{y}}\right]_{\underline{a}} \neq 0$
THE FINE PRINT: THE PARTIAL DERIVATIVES MUST EXIST AND BE CONTINUOUS
NOTE HOW THIS DERIVATIVE IS A SQUARE MATRIX...
THIS (LOCAL) SOLUTION IS UNIQUE AND DIFFERENTIABLE WITH
$\left[\frac{\partial \underline{y}}{\partial \underline{x}}\right]_{\underline{a}} = -\left[\frac{\partial F}{\partial \underline{y}}\right]_{\underline{a}}^{-1} \left[\frac{\partial F}{\partial \underline{x}}\right]_{\underline{a}}$
$\frac{dy}{dx} = -\left(\frac{\partial F}{\partial x}\right) \Big/ \left(\frac{\partial F}{\partial y}\right)$
COMPARE