MATH 223 Some Notes on Exam 3 of December 2021

1: We can describe D as $D = \{(x,y) : x^2 + y^2 \le 2^2\}$ or $D = \{(r,\theta) : 0 \le r \le 2, 0 \le \theta \le 2\pi\}$

$$\textbf{(a)} \int_{x=-2}^{x=2} \int_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}} (5 - x^2 - y^2)\, dy\, dx \quad \textbf{(b)} \int_{r=0}^{r=2} \int_{\theta=0}^{\theta=2\pi} (5 - r^2) r\, dr\, d\theta$$

(c): Parametrize γ by $\mathbf{g}(t) = (2\cos t, 2\sin t), 0 \le t \le 2\pi$. Then $\mathbf{g}'(t) = (-2\sin t, 2\cos t)$.
Now $\mathbf{F} = \nabla f = (f_x, f_y) = (-2x, -2y)$ so $\mathbf{F}(\mathbf{g}(t)) = (-2\cos t, -2\sin t)$.

$$\text{Thus } \int_\gamma \mathbf{F} = \int_0^{\pi/2} \mathbf{F}(\mathbf{g}(t)) \cdot \mathbf{g}'(t)\, dt = 2\int_0^{\pi/2} (4\sin t\cos t - 4\cos t\sin t)\, dt = 2\int_0^{\pi/2} 0\, dt = 0$$

(d): $\mathbf{F}$ is a gradient vector field so if γ is any path from (2,0) to (0,2), we have

$$\int_\gamma \mathbf{F} = f(2,0) - f(0,2) = (1 - 2^2 - 0^2) - (1 - 0^2 - 2^2) = 0$$

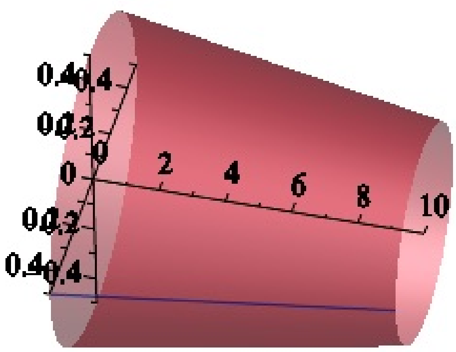
2: (There are several correct ways to do this problem). We will measure all distances in feet. The mass of an object is just the integral of the density function μ. As the density decreases from 50 to 40 as z increases from 0 to 10, we have $\mu(x,y,z) = 50 - z$. The cross section at height z is a circle of radius $\frac{1}{2} - \frac{z}{60}$ as the radius decreases linearly from 1/2 foot to 1/3 foot.
Using a triple integral employing cylindrical coordinates, the mass is

$$\int_{\theta=0}^{\theta=2\pi} \int_{z=0}^{z=10} \int_{r=0}^{r=\frac{1}{2}-\frac{z}{60}} (50 - z)\, r\, dr\, dz\, d\theta$$

You can also use a triple integral with cartesian coordinates:

$$\int_{z=0}^{z=10} \int_{x=-(\frac{1}{2}-\frac{z}{60})}^{x=(\frac{1}{2}-\frac{z}{60})} \cdot \int_{y=-\sqrt{(\frac{1}{2}-\frac{z}{60})^2-x^2}}^{y=\sqrt{(\frac{1}{2}-\frac{z}{60})^2-x^2}} (50 - z)\, dy\, dx\, dz$$

We can also obtain the solid by revolving about the x-axis the line segment from (0, 1/2) to (10, 1/3). The equation of the line is $y = f(x) = \frac{1}{2} - \frac{1}{60}x$ so the cross-section at x is a circle of radius $\frac{1}{2} - \frac{1}{60}x$ which has area $\pi\left(\frac{1}{2} - \frac{x}{60}\right)^2$.
The density μ at x is $\mu(x) = 50 - x$ since it decreases linearly from 50 at $x = 0$ to 40 at $x = 10$.
Hence the mass of a cross-section at x is $(50 - x)\pi\left(\frac{1}{2} - \frac{x}{60}\right)^2$ so total mass is $\int_0^{10} (50 - x)\pi\left(\frac{1}{2} - \frac{x}{60}\right)^2 dx$.



3(a):

$$f'(x) = \frac{d}{dx}\int_0^1 \frac{u^x - 1}{\ln u}\, du = \text{by Leibniz} \int_0^1 \frac{\partial}{\partial x}\frac{u^x - 1}{\ln u}\, du$$
$$= \int_0^1 \frac{u^x \ln u - 0}{\ln u}\, du = \int_0^1 u^x\, du = \left.\frac{u^{x+1}}{x+1}\right|_0^1 = \frac{1^{x+1} - 0^{x+1}}{x+1} = \frac{1}{x+1}$$

3(b): $f(x) = \int \frac{1}{x+1}\, dx = \ln|x+1| + C$. But $f(0) = \int_0^0 \frac{u^{x}-1}{\ln u}\, du = 0$
so $0 = f(0) = \ln|0 + 1| + C = \ln 1 + C = 0 + C = C$ making $C = 0$. Hence $f(x) = \ln|1 + x| = \ln 1 + x$ since $x > -1$.

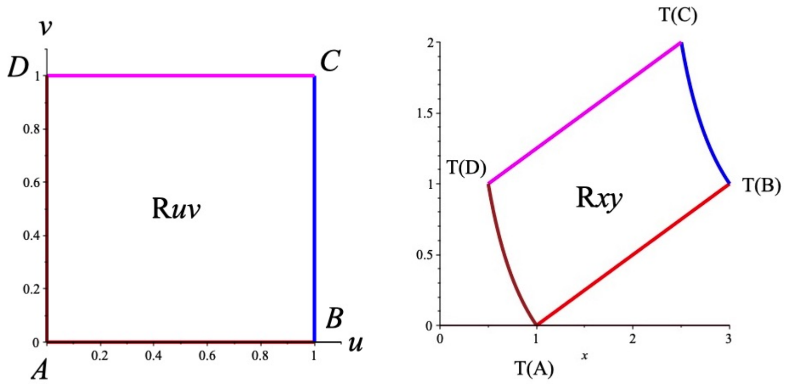
4: $\mathbf{T}(u,v) = \left(2u + \frac{1}{v+1}, u+v\right)$ has $\mathbf{T}'(u,v) = \begin{pmatrix} 2 & \frac{-1}{(v+1)^2} \\ 1 & 1 \end{pmatrix}$ so det $\mathbf{T}'(u,v) = 2 + \frac{1}{(v+1)^2}$ which is nonzero on R_{uv}. See below for a sketch of R_{xy}. To use Jacobi's Theorem. we must also show that $\mathbf{T}$ is one-to-one on the interior of R_{uv}. To that end, suppose $\mathbf{T}(u,v) = \mathbf{T}(w,z)$; then $2u + \frac{1}{v+1} = 2w + \frac{1}{z+1}$ and $u+v = w+z$. Write the last equation as $u - w = z - v = -(v-z)$ so $u-w$ is of the opposite sign as $v-z$) but the first equation gives

$$2u - 2w = \frac{1}{z+1} - \frac{1}{v+1} = \frac{(v+1)-(z+1)}{(z+1)(v+1)} = \frac{v-z}{(z+1)(v+1)}.$$

But $z+1 > 0$ and $v+1 > 0$on R_{uv} so $u-w$ has the same sign as $v-z$. Hence $u-w$ must be 0 and so $u = w$. But $z - v = u - w = 0$ so $v = z$ also.

By Jacobi's Theorem,

$$\begin{aligned}
\text{Area of } R_{xy} = \int_{R_{xy}} 1\,dx\,dy &= \int_{R_{uv}} det\mathbf{T}'\,dv\,du \\
&= \int_{R_{uv}} 2 + \frac{1}{(v+1)^2}\,dv\,du = \int_{u=0}^{u=1}\int_{v=0}^{v=1} 2 + \frac{1}{(v+1)^2}\,dv\,du \\
&= \int_{u=0}^{u=1} \left[2v - \frac{1}{v+1}\right]_{v=0}^{v=1} du = \int_{u=0}^{u=1} (2 - \frac{1}{2}) - (0-1)\,du \\
&= \int_{u=0}^{u=1} \frac{5}{2}\,du = \frac{5}{2}
\end{aligned}$$



On Segment AB: points are of the form $(u,0)$ for $0 \le u \le 1$ so $T(u,0) = (2u+1, u) = (x,y)$ so $u = \frac{x-1}{2}$ and $y = \frac{x-1}{2}, 1 \le x \le 3$.
On segment BC: points are of the form $(1,v)$ for $0 \le v \le 1$ so $T(1,v) = (2 + \frac{1}{v+1}, 1+v)$ so $\frac{1}{v+1} = x-2$ and $y = 1 + v = \frac{1}{x-2}, 5/2 \le x \le 3$.
On segment CD: points are of the form $(u,1)$ for $0 \le u \le 1$, so $T(u,1) = (2u+1/2, u+1)$ so $2u = x - 1/2$ and $u = x/2 - 1/4$ giving $y = x/2 - 1/4 + 1 = x/2 + 3/4$ for $1/2 \le x \le 1$
On segment DA: points are of the form $(0,v)$ for $0 \le v \le 1$ so $T(u,v) = \left(\frac{1}{v+1}, v\right)$ so $x = \frac{1}{v+1}$ making $v + 1 = \frac{1}{x}$; then $y = v = \frac{1}{x} - 1$ for $1/2 \le x \le 1$

5a: We have

$$\int_0^\infty e^{-aw}\,dw = \lim_{b\to\infty}\int_0^b e^{-aw}\,dw = \lim_{b\to\infty}\left[\frac{-1}{a}e^{-aw}\right]_0^b = \lim_{b\to\infty}\left(\frac{-1}{a}e^{-ab} - \frac{-1}{a}e^0\right) = \frac{1}{a} \text{ if } a > 0.$$

Thus we have $\int_0^\infty e^{-5x}\,dx = \frac{1}{5}$ and $\int_0^\infty e^{-4x}\,dx = \frac{1}{4}$. Thus

$$k\int_Q e^{-5x-4y}\,dy\,dx = k\int_{x=0}^\infty\int_{y=0}^\infty e^{-5x}e^{-4y}\,dy\,dx = k\int_{x=0}^\infty e^{-5x}\,dx\int_{y=0}^\infty e^{-4y}\,dy = k\frac{1}{5}\frac{1}{4} = \frac{k}{20}$$

so we need $k = 20$ to obtain a probability function.

5b: Here $Prob(x+y>3) = 1 - Prob(x+y \le 3) = 1 - \int_{x=0}^{x=3}\int_{y=0}^{y=3-x} p(x,y)\,dy\,dx$ which equals

$$1 - \int_{x=0}^{x=3}\int_{y=0}^{y=3-x} 20e^{-5x}e^{-4y}\,dy\,dx = 1 - \int_{x=0}^{x=3} -5e^{-5x}e^{-4y}\Big|_{y=0}^{=3-x} dx$$

$$= 1 + \int_{x=0}^{x=3} 5e^{-5x}\left(e^{-12+4x} - e^{0}\right)\,dx = 1 + 5\int_{0}^{3}\left(e^{-12-x} - e^{-5x}\right)\,dx$$

$$= 1 + 5\left[-e^{-12-x} + \frac{1}{5}e^{-5x}\right]_{0}^{3} = 1 + 5\left[(-e^{-15} + \frac{1}{5}e^{-15}) - (-e^{-12} + \frac{1}{5})\right]$$

$$= 1 + 5\left[-\frac{4}{5}e^{-15} + e^{-12} - \frac{1}{5}\right] = 1 - 4e^{-15} + 5e^{-12} - 1 = 5e^{-12} - 4e^{-15}$$

6a: $\mathbf{g}(t) = (3t^2, 4t^3, -3t^4)$ gives $\mathbf{g}'(t) = (6t, 12t^2, -12t^3) = 6(t, 2t^2, -2t^3)$ with
$|\mathbf{g}'(t)| = 6\sqrt{(t^2 + 4t^4 + 4t^6)} = 6t\sqrt{1 + 4t^2 + 4t^4} = 6t\sqrt{(1+2t^2)^2} = 6t(1+2t^2) = 6t + 12t^3$
so length $= \int_0^2 |\mathbf{g}'(t)|\,dt = \int_0^2 (6t + 12t^3)\,dt = (3t^2 + 3t^4)\Big|_0^2 = 12 + 48 = 60.$

6b: (i) $\mathbf{g}(t)$ is position so $\mathbf{g}'(t)$ is velocity and $|\mathbf{g}'(t)|$ is speed so $s(t) = |\mathbf{g}'(t)|$

(ii): Since γ is a flow line for $\mathbf{F}$, we have $\mathbf{F}(\mathbf{g}(t)) = \mathbf{g}'(t)$ for all t in [0,1]. Now

$$\int_\gamma \mathbf{F} = \int_0^1 \mathbf{F}(\mathbf{g}(t))\cdot\mathbf{g}'(t)\,dt = \int_0^1 \mathbf{g}'(t)\cdot\mathbf{g}'(t)\,dt = \int_0^1 |\mathbf{g}'(t)^2|\,dt = \int_0^1 s^2(t)\,dt$$

The first equality coming from the definition of line integral and the second equality from the fact that γ is a flow line for $\mathbf{F}$. Thus the two integrals are equal.

7a: The equality $f(x,y) - f(a,y) = \int_a^x f_x(t,y)\,dt$ comes from the Fundamental Theorem of Calculus.

7b: Differentiate the equation from (a) with respect to y: $f_y(x,y) - f_y(a,y) = \frac{\partial}{\partial y}\int_a^x f_x(t,y)\,dt$. Applying Leibniz's Rule to the right hand side: $\frac{\partial}{\partial y}\int_a^x f_x(t,y)\,dt = \int_a^x \frac{\partial}{\partial y} f_x(t,y)\,dt = \int_a^x f_{xy}(t,y)\,dt$.
Thus $f_y(x,y) - f_y(a,y) = \int_a^x f_{xy}(t,y)\,dt$.
Differentiate this last identity with respect to x: $f_{yx}(x,y) - 0 = f_{xy}(x,y)$ using the Fundamental Theorem of Calculus again.