MATH 223 Multivariable Calculus

Some Notes on Sample Examination 1

1. An object moves in space in such a way that its position *f(t)* at each time *t* is given by the vector-valued function $\boldsymbol{f}(t) = (\cos t, \ln(1+t^2),\ e^{-2t})$

Compute each of the following

(a) $\boldsymbol{f}'(t) = (-\sin t, \frac{2t}{1+t^2}, -2e^{-2t})$

(b) $\boldsymbol{f}''(t) = (-\sin\ t, \frac{2(1-t^2)}{(1+t^2)^{2,}}, 4\, e^{-2t})$

(c) Position at $t = 0$: $\boldsymbol{f}(0) = (\cos 0, \ln(1+0^2),\ e^{-2*0}) = (1,0,1)$

(d) Velocity at $t = 0$: $\boldsymbol{f}'(0) = (0,0,-2)$

(e) Speed at $t = 0$: $|\ \boldsymbol{f}'(0)| = \sqrt{0^2+0^2+(-2)^2 = }\,2$

(f) Parametric equation for the tangent line to the curve at $t = 0$:
Solution: (1,0,1) + (0,0,-2)*t*

(g) The dimension of the image of *f*. The image is a curve and hence, has dimension 1.

2. An object moves in space in such a way that its position is given by some twice differentiable vector-valued function **p**(t) in such a way that its *speed* has a constant value of 4. Show that velocity and acceleration vectors are always orthogonal.

Solution: We have | **v** | = 4 and hence **v • v = | v |²** = 16. Differentiating each side with respect to *t* yields 0 = (**v • v)' = v • v'+ v' • v = 2 v • v'** so **v • v' = v • a** where **a** is the acceleration vector. Since the dot product of velocity and acceleration is 0, the two vectors are orthogonal.

3. Let g: $\mathbb{R}^2 \to \mathbb{R}^1$ be given by $g(x,y) = 2x^2 - 3y^2$. Sketch the level curves in $\mathbb{R}^2$ for

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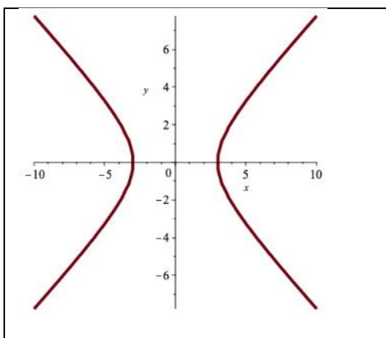
<td>
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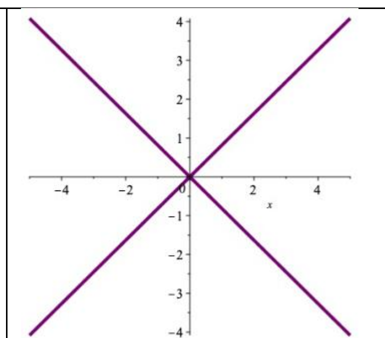
<td>
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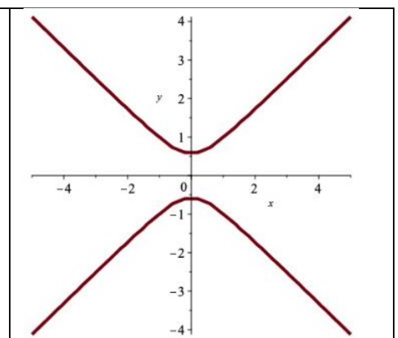
<td>
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<td>(a) g(x,y) =18</td>
<td>(b) g(x,y) =0: $2x^2 - 3y^2 = 0$ is $y^2 = \frac{2}{3}x^2$. So $y = \pm\sqrt{\frac{2}{3}}\ x$</td>
<td>(c) g(x,y) =-1</td>
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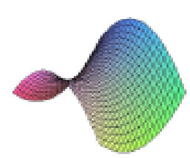

Circle the picture below which best represents the graph of *g*:

4. In 1938, two future winners of the Nobel Prize in Economics, Ragnar Frisch and Trygve Haavelmo, published a paper "Ettersporselen etter melk i Norge" ("The Demand for Milk in Norway") They found that the milk production z is related to p, the relative price of milk, and r, the income per family through the equation$z = f(r,p) = k\frac{r^a}{p^b}$ where k, a and b are positive constants. Suppose $b = 3/2$ and $a = 2$.

(a) Use the ***definition*** of partial derivatives to find $\frac{\partial z}{\partial r} = f_r$ at the point where $r = 3$, $p = 4$

Solution: $f(3+t,4) - f(3,4) = k\,\frac{(3+t)^2}{4^{\frac{3}{2}}} - k\,\frac{3^2}{4^{\frac{3}{2}}} = \frac{k}{4^{\frac{3}{2}}}[3^2 + 6t + t^2 - 3^2] = \frac{k}{4^{\frac{3}{2}}}[6t + t^2]$

So $\frac{f(3+t,4)-f(3,4)}{t} = \frac{k}{8}\left(\frac{6t+t^2}{t}\right) = \frac{k}{8}(6+t)$ [since $4^{\frac{3}{2}} = 8$] which approaches $\frac{6k}{8} = \frac{3k}{4}$

This $f_r(3,4) = \lim_{t\to 0}\frac{f(3+t,4)-f(3,4)}{t} = \frac{3k}{4}$

(b) Find gradient $\nabla f(x,y)$ if $f(x,y) = k\,\frac{x^2}{y^{3/2}}$ for some constant k.

Solution: $\nabla f(x,y) = \left(f_x(x,y), f_y(x,y)\right) = \left(\frac{2kx}{y^{3/2}}, \frac{-3kx^2}{2y^{5/2}}\right)$

(c) Find an equation for the tangent plane at (3, 4, f(3,4)) to the surface $z = f(x,y)$

Solution: $z = \frac{9k}{8} + \frac{3k}{4}(x-3) - \frac{27k}{64}\,(y-4)$.

5. Let f be the real-valued function of two variables defined by

$$f(x,y) = \begin{cases} \dfrac{xy}{ax^2 + by^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

where a is number of the month and b is the day on the month on which you were born. For example, if your birthday is October 4, you would use $a = 10$, $b = 4$ for the rest of this problem.

(a) Find the limit of f as (x,y) approaches (0,0) along the line $y = x$.

$Solution$: $f(x,x) = \frac{xx}{ax^2+bx^2} = \frac{1}{a+b}$ so limit is $\frac{1}{a+b}$

(b) Find the limit of f as (x,y) approaches (0,0) along the line $y = -x$.

$Solution$: $f(x,-x) = \frac{(x)(-x)}{ax^2+bx^2} = \frac{-1}{a+b}$ so limit is $\frac{-1}{a+b}$

(c) Prove that $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Solution: Since $a + b$ is always positive, we get two different limits along the two different lines. One is positive and one is negative. Since there is disagreement on a limiting value, there is no overall limit as (x,y) approaches the origin.

(d) Find the maximum value **M** of *f(x,y)*. *Solution*: Since f has value 0 along the vertical axis, and is positive along the line $y = x$, the maximum is a positive number which lies along the line $y=mx$ for some constant m. Now $f(x,mx) = \frac{xmx}{ax^2+bm^2x^2} = \frac{m}{a+bm^2}$. . This fraction has value k when $k(a+bm^2) = m$ which we can rewrite as $bk\,m^2 - m + ak = 0$. By the quadratic formula, this equation has a real root exactly when $1 - 4abk^2 \geq 0$ which occurs when $k^2 \leq \frac{1}{4ab}$. Thus the largest possible value for k is $\frac{1}{2\sqrt{ab}}$ which is the maximum value **M.**

(e) What subset of the real numbers is the image of this function?

Solution: The closed interval $\left[-\frac{1}{2\sqrt{ab}}, \frac{1}{2\sqrt{ab}}\right]$

6. (a) Find the functions f_{xy}, $f_{zy,}$ and $f_{xyz,}$ if $f(x,y,z) = \frac{x^2y}{z}$.

$f_x(x,y,z) = \frac{2xy}{z}, f_y(x,y,z) = \frac{x^2}{z}, f_z(x,y,z) = -\frac{x^2y}{z^2}$

Then $f_{xy} = (f_x)_y$ so $f_{xy}(x,y) = \left(\frac{2xy}{z}\right)_y = \frac{2x}{z}$

and $f_{zy} = (f_z)_y$ so $f_{zy}(x,y) = \left(\frac{-x^2y}{z^2}\right)_y = \frac{-x^2}{z^2}$

while $f_{xyz} = (f_{xy})_z = \left(\frac{2x}{z}\right)_z = \frac{-2x}{z^2}$

(b). Show that the function given by $f(s,t) = (s\cos t, s\sin t, s)$ for $0 \le s \le 4$, $0 \le t \le 2\pi$ twists a rectangle in the (s,t)-plane into a piece of the surface in 3-dimensional space satisfying the equation $x^2 + y^2 = z^2$. Sketch and describe what that surface looks like. Provide a clear explanation of your reasoning.

Solution: $x^2 + y^2 = s^2cos^2t + s^2sin^2t = s^2 = z^2$

z-slice : circle with radius |*z*|, centered on *z*-axis above origin in (*x*,*y*)-plane. *x*-slices and *y*-slices are pairs of lines. The surface is a circular cone with vertex at the origin (0,0,0)