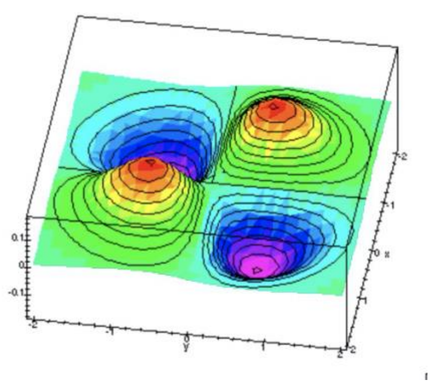
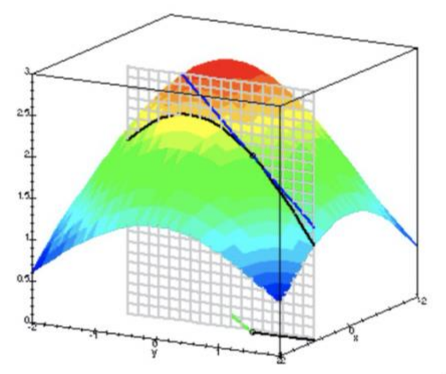
MATH 223

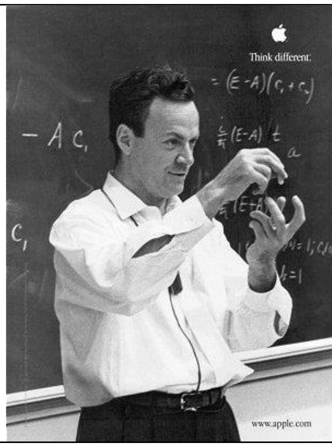
Multivariable Calculus





Fall 2022

Michael Olinick



Richard Feynman

May 11, 1918 – February 15, 1988

Nobel Prize in Physics, 1965

“If you are interested in the ultimate character of the physical world, or the complete world, and at the present time our only way to understand that is through a mathematical type of reasoning, then I don't think a person can fully appreciate, or in fact can appreciate much of, these particular aspects of the world, the great depth of character of the universality of the laws, the relationship of things, without an understanding of mathematics. I don't know any other way to do it, we don't know any other way to describe it accurately... or to see the interrelationships without it. So I don't think a person who hasn't developed some mathematical sense is capable of fully appreciating this aspect of the world — don't misunderstand me, there are many, many aspects of the world that mathematics is unnecessary for, such as love, which are very delightful and wonderful to appreciate and feel awed and mysterious about; and I don't mean to say that the only thing in the world is physics, but you were talking about physics and if that's what you're talking about, then to not know mathematics is a severe limitation in understanding the world."

—Richard Feynman (Nobel Prize in Physics, 1965),

The Pleasure of Finding Things Out


Contents

MATH 223 : Multivariable Calculus
Course Description/Syllabus
Fall Term 2022

***Course Title*:** **Multivariable Calculus**

***Catalog Description*:** The calculus of functions of more than one variable. Introductory vector analysis, analytic geometry of three dimensions, partial differentiation, multiple integration, line integrals, elementary vector field theory, and applications (formerly MA 201).

***Additional Description from Mathematics Department Webpage*:** All the functions you've studied in calculus so far live on a flat piece of paper. But you live in (at least!) three dimensions. Now you certainly know that calculus was invented to solve problems about the physical world, so we're going to have to move off that flat paper at some point. MATH 223 is where it happens. The key is the concept of a **vector**. If you've had a little bit of physics, you may have heard a vector is an object having direction and magnitude. In MATH 223, we'll tighten that definition up, and study functions whose domains and ranges consist of vectors. Can limit, derivative and integral make sense out here? The answer is yes, and when you're through you'll know how Newton's calculus – the greatest intellectual achievement of humankind! – made sense of Kepler's empirical observations about the motion of the planets – the greatest scientific discovery of all time! Come to think of it, maybe this course should be required for graduation…

Course Website: **f22.middlebury.edu/MATH0223a**

***Instructor*:** Michael Olinick, Office: Warner 202, Phone: 443-5559. Home telephone: 388-4290; email: *molinick@middlebury.edu*. Usual Office Hours: Monday, Wednesday and Friday from 12:15 to 2:30 PM in Room 206 of the 75 Shannon Street building. I would be happy to make an appointment to see you at other mutually convenient times. Due to Covid restrictions and security concerns, we are unable, at this time, to meet with students on the first floor of the Shannon Street building. We may have to schedule some meetings on Zoom.

***Meeting Times*:** MATH 223A: MWF 10:10 AM – 11:00 AM (Warner 101)

***Prerequisites*:** Calculus II (MATH 0122) **and** Linear Algebra (MATH 200) or permission.

***Textbook*:** Michael Olinick, *Multivariable Calculus: A Linear Algebra Based Approach*, First Edition: Kendall-Hunt, 2022; ISBN 9781792437915 (https://he.kendallhunt.com/product/multivariable-calculus-linear-algebra-based-approach)

Your daily assignments will include a few pages of reading in the text. Be certain to read the book carefully (with pencil and paper close by!) and to complete the relevant reading **before** coming to class and before embarking on the homework problems.

Supplemental Book: James A. Carlson and Jennifer M. Johnson, *Multivariable Mathematics with Maple,* (Prentice-Hall, 1997). We will distribute portions in class.

Computer Algebra Systems: Mathematically oriented software such as *Maple*, *MATLAB*, and *Mathematica* give you an opportunity to investigate the ideas of multivariable calculus in ways not available to previous generations of students. Relatively simple commands can direct a computer to carry out complex calculations rapidly and without error. More importantly, you can create and carry out experiments to develop and test your own conjectures. The very powerful graphics capabilities of these applications provide you with strong tools to deepen your understanding of multivariable calculus through visualization of curves and surfaces. I will feature *Maple* in my presentations but you should feel free to *MATLAB* or *Mathematica* or other computer algebra systems. I will try to schedule a few optional introductory *Maple* lab sessions.

***Requirements*:** There will be three midterm examinations and a final examination in addition to required daily homework assignments and an extended independent project. The midterm examinations will be given in the evening to eliminate time pressure. Tentative dates for these tests are:

Monday, October 3
Wednesday, November 2
Wednesday, November 30

Final Exam: The registrar's office has set these days and times for the final exam:

MATH 223A: Thursday, May 19 from 9 AM to 12 Noon.

You will likely be able to take the final at either time, but will need to let me know by Monday, May 16.

Course Grades: Each of the midterm exams will be worth approximately 20%, the final about 25% and the independent project and homework roughly 15%. I will make adjustments with later work counting more heavily if students show improvement over earlier results.

Accommodations Students who have Letters of Accommodation in this class are encouraged to contact me as early in the semester as possible to ensure that such accommodations are implemented in a timely fashion. For those without Letters of Accommodation, assistance is available to eligible students through the Disability Resource Center (DRC). Please contact ADA Coordinators Jodi Litchfield and Peter Ploegman of the DRC at ada@middlebury.edu for more information. All discussions will remain confidential.

Homework: Mathematics is not a spectator sport! You must be a participant. The only effective way to learn mathematics is to do mathematics. In your case, this includes working out many multivariable calculus problems.

There will be daily written homework assignments which you will be expected to complete and submit. They will be corrected and assigned a numerical score, but I view these assignments primarily as **learning** rather than testing experiences. I will occasionally assign some challenging problems which everyone may not be able to solve. You should, however, make an honest attempt at every problem.

Each homework assignment will probably take you between 2 and 3 hours to complete; this includes the reading and problem solving. If you keep pace with the course by spending an hour or so each day on it, then you will be quite successful. If you wait until the end of the week and then try to spend one six hour block of time on the material, then experience shows you face disaster!

Grades: Grades in the course will be based primarily on the examinations, homework, projects, lab reports, and class participation.

Help: Please see me immediately if you have any difficulties with this course. There are ample resources on campus for assistance. Our course tutor is Abby Truex, ’04. She will soon post hours when she is available for help.

One of the essential characteristics of college life that distinguishes it from secondary school is the increased responsibility placed on *you* for your own education. **Most of what you will learn will not be told to you by a teacher inside a classroom**.

Even if our model of you were an empty vessel waiting passively to be filled with information and wisdom, there would not be time enough in our daily meetings to present and explain it all.

We see you, more appropriately, as an *active* learner ready to confront aggressively the often times subtle and difficult ideas our courses contain. You will need to listen and to read carefully, to master concepts by wrestling with numerous examples and problems, and to ask thoughtful questions.

As you progress through the undergraduate mathematics curriculum, emphasis changes from mastering techniques to solve problems to learning the theory that underlies the particular subject you are studying. *Multivariable Calculus* is a transitional course. You will do plenty of calculations, find many derivatives and deal with a full quota of integrals. You will also find more of your effort directed toward understanding definitions, statements of theorems and their proofs. You will even be expected to come up with some short proofs of your own.

One of my goals for you this term is to develop your skills in *reading* mathematical expositions. I will expect that you will have read (perhaps more than once!) in advance the sections of the text relevant to the topic we will be exploring in class that day. I will not normally present a lecture which substitutes for reading the text. I will more likely use time in class to give a broader overview or alternative proofs or interesting applications and extensions of the material or previews of the next section.

MATH 223: Fall, 2022
Tentative Course Outline
(Times are approximate)

I. Review (on your own as needed)
Single Variable Calculus
Vectors
Equations and Matrices
Vector Spaces and Linearity

II. Derivatives (2 weeks)
Functions of One Variable
Several Independent Variables
Partial Derivatives
Parametrized Surfaces

III. Differentiability (1 week)
Limits and Continuity
Real-Valued Functions
Directional Derivatives
Vector-Valued Functions

IV. Vector Differential Calculus (2+ weeks)
Gradient Fields
The Chain Rule
Implicit Differentiation
Extreme Values
Curvilinear Coordinates

V. Multiple Integration (3 weeks)
Iterated Integrals
Multiple Integrals
Integration Theorems
Change of Variable
Improper Integrals

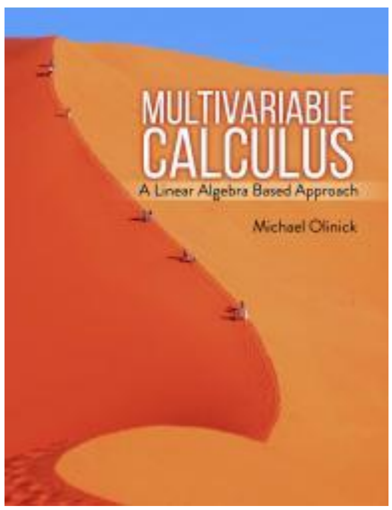
VI. Integrals and Derivatives on Curves (1 week)
Line Integrals
Weighted Curves and Surfaces of Revolution
Normal Vectors and Curvature
Flow Lines, Divergence, and Curl

VII. Vector Field Theory (2+ weeks)
Green's Theorem
Conservative Vector Fields
Surface Integrals
Gauss's Theorem
Stokes's Theorem

MATH 223 Multivariable Calculus
Guide To Assignments for Fall 2022

Each assignment is due in class on Monday, Wednesday, and Friday. Homework will generally focus on material introduced in the previous class meeting. Begin the new assignment as soon after class as practical so that you will have two days to complete it. Review your class notes and read the assigned sections first, and then work the problems. We encourage you to study together and wrestle with hard problems, but submitted solutions should be your own work. Come to class with well-prepared and specific questions on the assignment.

Experience shows that doing the assignments regularly and carefully is the key to doing well in calculus. You should expect to spend at least two hours on homework for every hour in class, and an average of 8 to 10 hours total each week including your review and careful reading. Most of the readings and exercises will come from our text, of *Multivariable Calculus: A Linear Algebra Based Approach* (https://he.kendallhunt.com/product/multivariable-calculus-linear-algebra-based-approach)



The reading is an important supplement to what goes on in class. It will cover some examples and methods you need and for which you will be held responsible but which we will not have time to cover in class. Read carefully with a pencil and pad next to you. Work out details of calculations you don't understand.

Some Applications Related To The Material of This Course

Below are a dozen or so exercises that show some of the range of applications of *Multivariable Calculus*. You should be able to solve these problems and similar ones by the end of the course. If applications like these appeal to you, then you will enjoy applying the tools and techniques you will be learning in our class.

1. **Temperature Lapses.** The normal lapse rate for temperature above the surface of the earth assumes a steady drop of 3ºF per 1000 feet of increase of elevation. Under this assumption, with ground temperature 32ºF, and assuming negligible air resistance, estimate the temperature at time t at the height of a projectile fired straight up with an initial speed of 300 feet per second. What is the minimum temperature attained?

2. **Airplane Headings.** An airplane pilot wishes to maintain a true course in the direction 240º with a ground speed of 400 miles per hour when the wind is blowing directly north at 50 miles per hour. Find the required airspeed and compass heading.

3. **Big Bertha.** In World War I, Paris was bombarded by guns from the unprecedented distance of 75 miles away, shells taking 186 seconds to complete their trajectories. Estimate the angle of elevation at which the gun was fired and the maximum height of the trajectory, assuming negligible air resistance. (During a substantial part of the trajectory the altitude was high enough that air resistance was negligible there).

4. **Fox and Rabbit.** Suppose that a rabbit runs with constant speed $v > 0$ on a circular path of radius a, and that a fox, also running with constant speed v, pursues the rabbit by starting at the center of the circle, always maintaining a position on the radius from the center to the rabbit. How long does it take the fox to catch the rabbit? What is the path of motion of the fox?

5. **Heat Loss** A rectangular building is to be built to contain a fixed volume V. Heat loss through the roof and walls is proportional to area, while heat loss through the floor is negligible. Heat loss through the roof material is 3 times as rapid as heat loss through the wall material. What dimensions will minimize heat loss?

6. **Ideal Gas Law.** According to the ideal gas law, the pressure P, volume V and temperature T of a confined gas are related by the formula $PV = kT$ for a constant k. Express P as a function of V and T, and describe the level curves associated with this function. What is the physical significance of these level curves?

7. **Drug Use.** If a drug is taken orally, the time T at which the largest amount of drug is in the bloodstream can be calculated using the half-life x of the drug in the stomach and the half-life y of the drug in the bloodstream. For many common drugs (such as penicillin), T is given by $T = \dfrac{xy(\ln x - \ln y)}{(x - y)\ln 2}$. For a certain drug, x = 30 minutes and y = 1 hour. If the maximum error in estimating each half-life is ±10%, find the maximum error in the calculated value of T.

8. **A Toddler's Surface Area.** At age 2 years, a typical boy is 86 cm tall, weighs 13 kg and is growing at a ate of 9 cm/year and 2 kg/year. Use the Dubois and DuBois surface area formula $S = 0.007184x^{0.425}y^{0.725}$ for weight *x* and height *y* to estimate the rate at which the body surface area is growing.

9. **Hardy-Weinberg** Three alleles (alternative forms of a gene) A, B and O determine the four human blood types: A (AA or AO), B (BB or BO), O (OO) and AB. The *Hardy-Weinberg* law asserts that the proportion of individuals in a population who carry two different alleles is given by the formula $P = 2pq + 2pr + 2rq$ where *p, q,* and *r* are the proportions of alleles A, B, and O, respectively, in the population. Show that P must be les than or equal to 2/3

10. **Cobb-Douglas Production Functions** If *x* units of capital and *y* units of labor are required to manufacture *f(x,y)* units of a certain commodity, the Cobb-Douglas production function is defined by $f(x,y) = kx^a y^b$ where *k* is a constant and *a* and *b* are positive numbers *a* + *b* = *1*. Suppose that $f(x,y) = x^{1/5}y^{4/5}$ and that each unit of capital costs *C* dollars and each unit of labor costs *L* dollars. If the total amount available for these costs is *M* dollars, *so xC* + *yL* = *M*, how many units of capital and labor will maximize production?

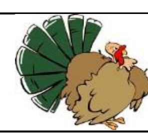
11. **Volume of the earth.** Because of rotation, the earth is not perfectly spherical bt is slightly flattened at the poles, with a polar radius of 6356 km and an equatorial radius of 6378 km. As a result, the shape of the earth's surface can be approximated by the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ with *a* = *b* = 6378 and *c* = 6356. Estimate the volume of the earth.

12. **Gypsy Moths on the Loose.** We can model the spread of an aerosol in the atmosphere by a combination of wind and diffusion using a function of three variables. The concentration of the aerosol reaches a steady state independent of time at a point **x** given by $S(\mathbf{x}) = \dfrac{Q}{2\pi v_0 \sigma_y \sigma_z x^{2-n}} e^{-\left(y^2/\sigma_y^2 + z^2/\sigma_z^2\right)/2x^{2-n}}$ if the aerosol is released from the origin and the wind is blowing in the positive *x*-direction with velocity v_0. Here Q is the rate of release of the chemical and σ_y, σ_z and n are empirically determined constants. We measure distance in centimeters, velocity in centimeters per second, and concentration in parts per cubic centimeter. The values $\sigma_y = 0.4$, $\sigma_z = 0.2$ and $n = 0.25$ give good results for wind velocities less than 500 cm/s. Among other applications, we can model the diffusion of insect pheromones.

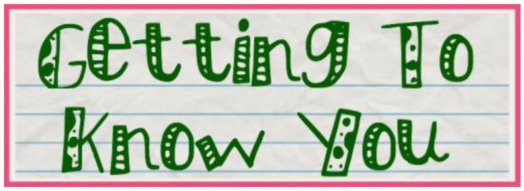
Pheromones are "odor" chemicals released by animals for chemical communication within a species. For a single female gypsy moth, Q is on the order of 3×10^{13} particles per cubic centimeter. Given that a male gypsy moth can detect as few as 10 0 particles per cubic cm, use the Lagrange multipliers method to determine the maximum distance downward from a female moth that a male moth can detect a female gypsy moth.

13. **Optimal Investment Policy.** An organization wants to determine an optimal allocation of investment between labor and capital stock over an extended period of time, beginning at time $t = 0$ and ending at time $t = T$ (which may be infinite). If s and e represent the fraction of the output allocated to investment in capital stock and human labor, respectively, then we want to choose s and e so that we maximize $\int_O^T (1 - s - e)\, wf\, e^{-\gamma t}\, dt$. Here w represents fractional employment, $f(r/w)$ is the output per employed worker, r is the capital-labor ratio, and γ is the consumption rate. Note that we require $r'(t) = swf - (n+\delta)r$ where n is the rate of growth of population and δ is the rate of depreciation of capital and we also need $w'(t) = (e/n)wf - (n+\mu)w$ where μ is the death rate. Use Stokes's Theorem to find the optimal allocation plan.

MATH 223: Multivariable Calculus
Tentative Schedule for Fall Term 2022

Week Of:	Monday	Tuesday	Wednesday	Thursday	Friday
September 12	*FIRST DAY OF CLASSES*		Assignment 1		Assignment 2
September 19	Assignment 3		Assignment 4		Assignment 5
September 26	Assignment 6		Assignment 7		Assignment 8
October 3	Assignment 9 **Exam 1**				Assignment 10
October 10	Assignment 11		Assignment 12		***MID TERM RECESS***
October 17	Assignment 13		Assignment 14		Assignment 15
October 24	Assignment 16		Assignment 17		Assignment 18
October 31	Assignment 19		Assignment 20 **Exam 2**		Assignment 21
November 7	Assignment 22		Assignment 23		Assignment 24
November 14	Assignment 25		Assignment 26		Assignment 27
November 21	**Thanksgiving Recess**		**Thanksgiving Recess**		**Thanksgiving Recess**
November 28	Assignment 28		Assignment 29 **Exam 3**		Assignment 30
December 5	Assignment 31		Assignment 32		Assignment 33
December 12	*Last Class*				

MATH 223 Fall 2022
Assignment 1
(Adapted from *Homework 0* by Alex Lyford)
Due: Wednesday, September 14



Reading

Read carefully Section 2.1 “Curves in the Plane and Space” in our text *Multivariable Calculus: A Linear Algebra Based Approach.*

Writing

You may submit an electronic copy of this assignment to me (molinick@middlebury.edu) with the subject line: **MATH 223 Assignment 1** or print it out and bring it to Wednesday’s class. Make sure you include your name at the top of the document.

Your task is to create a document describing yourself, your goals, and what you hope to get out of our Multivariable Calculus class. Please provide your name at the top of the first page along with your major or likely major and your anticipated graduate date.

Start with an autobiographical statement about yourself that will help me to get to know a little about you. Where did you grow up? Why did you come to Middlebury? What are your likes and dislikes? Do you have any hobbies that you do regularly? Do you have a major extracurricular activity such as athletics, theatre or *The Campus?*

After the biographical statement tell me about your mathematical, statistical, and computer programming background. What did you like about previous mathematics, statistics, and/or programming classes? What did you dislike? What aspects did you find easy? What aspects did you find challenging?

The next part should discuss your plans for the remainder of your time in college, and what you hope to do after you graduate. Is more schooling the next step, or do you plan to get a job? It's okay to not have any idea what you want to do after graduation, but list some possibilities so that I can better tailor the materials in class to your potential career opportunities.

Finally tell me about your thoughts and expectations for this class. What are you hoping and/or expecting to learn? What do you think the challenges of this course might be? What, if anything, have you heard about this course from your peers? What expectations do you have of me? Feel free to also discuss anything I've failed to ask here.

Advice on Reading Your Mathematics Textbooks

[There are many excellent essays online with advice on how to read effectively a mathematics book. Here is an adaption of one of them. I haven't been able to track down the original author's name to give full credit.]

Reading the textbook is important for succeeding academically, and this is also true in your mathematics classes. However, reading mathematics is different from other types of reading. Getting the most out of a mathematics textbook will require more than just skimming through the pages. Below are some tips for helping you get the most from your mathematics text.

• **Focus on concepts, not exercises**. The most important material in a mathematics textbook is found in the prose, not in the exercises at the end of the section. In the past, you may have opened your mathematics book only when doing problem sets and exercises (looking at the rest of the book only for examples which mirror the current assigned homework). You must rid yourself of this bad habit now. Instead, set aside time to read the text when you are not working on a homework assignment. This will enable you to truly focus on the mathematical concepts at hand.

There are an infinite number of types of mathematics problems, so there is no way to learn every single problem-solving technique. Mathematics is about ideas. The mathematics problems that you are assigned are expressions of these ideas. If you can learn the key concepts, you will be able to solve any type of problem (including ones you have never seen before) that involves those concepts.

• **Read the text more than once.** You cannot read mathematics in the same way as you would read a newspaper or a novel. Many of the ideas presented in a typical college mathematics course have confounded brilliant minds in centuries past. So it is not unexpected that you may have difficulty learning these same ideas if you quickly scan through the reading assignments only once. You should expect to go through the each reading assignment several times before you can gain a full understanding of the material.

• **When reading through for the first time, look for the big ideas**. The first time you read through a chapter of the textbook, you should be thinking to yourself: “What is the main point of the chapter?” Look for the big picture. The details are important, but you need to be aware of the forest first before focusing on the trees.

• **The second time through, fill in details**. After you get the big picture, you should then look at the details. Take some time to think about each of the definitions, theorems, and formulas you encounter (more on this later).

• **Read with paper and pen**. As you are reading through the text, you should be writing notes and verifying any parts of which you are skeptical. Check any calculations. Rewrite definitions and theorems in your own words.

See if you can come up with your own examples. Ask yourself about special cases of the theorems you read.

- **Read the narrative**. There is a story to be told in mathematics. What is the progression of ideas being told? Don't just skip to the formulas and examples, but instead follow the development of the ideas and concepts presented.

- **Study the examples**. What points do each of the examples illustrate? Some examples are extreme cases. Other examples are supposed to illustrate "typical" situations.

- **Read the pictures**. There are good reasons for the many pictures and graphs in mathematics texts. You should be asking yourself what features of the picture are important to the key concepts. Focus on how each picture illustrates a particular idea.

- **Learn the vocabulary and the language.** Pay attention to definitions and what they mean. Mathematics language is very precise, and a word in a mathematical context may have a different meaning than when it is used in everyday conversation. In mathematics, great care is taken to explicitly and precisely define the notions being considered. In addition, mathematical definitions and language are crafted in such a way to convey sophisticated notions in as simple and concise a manner as possible.

- **Learn the theorems and what they mean.** Theorems are vital bricks to building mathematical knowledge. When you see a theorem in a mathematics text, look at it very closely. What does it say? What are its hypotheses? What implications does it have? Are there special cases you should be aware of ? Can you think of examples to which the theorem applies? Can you think of examples that do not satisfy the hypotheses and the conclusion of the theorem?

- **Use the index and the appendices**. Know what every word means. Make sure that you understand all of the words and ideas. If there is a particular word which you do not know (or which you want to know better), look it up. Use the table of contents or the index to help you.

- **Make a note of things you do not understand, and ask for help afterwards.** Even after following all of the above advice, you might still find some of the ideas confusing. That is to be expected; material such as this is often hard to internalize when one first encounters it. If there is something that you do not understand, make a note of it. Write down any questions you may have. You then can bring up these issues with your instructor or a classmate.

ON STUDYING AND LEARNING MATHEMATICS

Past students have found that some ways are far more effective than others in studying and learning mathematics. Here are some suggestions and pointers that may help you in budgeting the time you can devote to mathematics, preparing for examinations, and learning and understanding the material in a way that promotes long-range retention:

1. Do all reading assignments actively. Keep a pencil and scratch paper at hand. Mark up the pages of the book. Write in any questions you may have. Verify examples given by writing out the details yourself.

2. Plan to do all reading assignments several times. In mathematics courses, reading assignments are seldom more than a few pages long. They often contain, however, subtle ideas which require repeated study before they are mastered. You should read the appropriate section of the text before the class in which it will be discussed, read it again before beginning the homework assignment, and read it a third time after you have completed the homework.

3. Follow the advice in (1) above when reviewing your lecture notes. You should try to go over your lecture notes as soon as possible after the class session has ended. Definitely review the notes before attempting the homework.

4. Do all homework sets on time. Don't let yourself fall behind. If you have difficulty with a problem, especially one that is more theoretical, do the following:

   (a) Write out the relevant definitions and results. It may now be a small step to complete the problem.

   (b) Ask whether you can think of a simpler but related problem, and tackle that one first. Is there a special case of the general result? Do you know how to solve the problem in this special case? This approach usually provides insight for attacking the original problem.

5. Do not spend hours sitting still, thinking, reading, studying and reviewing problem solutions! While these approaches may be helpful for other courses and some time should be spent on these activities in mathematics courses, there are more productive paths to learning in mathematics. Spend your time writing out solutions to new problems, deriving relationships, writing down clear definitions, and outlining the steps of a proof. These activities provide a better way to prepare for an examination.

6. Pay a great deal of attention to definitions. Write them out yourself and think about them. Write out examples that do and do not satisfy the definitions. Ask yourself how the definition says something different from its intended meaning if the order of the words is shifted.

7. Begin reviewing for examinations a week early. Use small chunks of time. Tackle those topics you have found difficult; with hindsight they are often easier. Do NOT plan on spending a whole day of study just before an exam. This is almost always an inefficient way to budget your time.

8. Review solutions for homework problems as soon as you get them, and write up (for your own enlightenment) those problems which caused you difficulty.

9. Write down questions that arise as you go along. Bring them with you to class, to review sessions, and to your instructor's office hours.

ON PROBLEM SOLVING

A major part of your time in Multivariable Calculus and other courses is devoted to solving problems. It is worth your while to develop sound techniques. Here are a few suggestions.

Think. Before plunging into a problem, take a moment to think. Read the problem again. Think about it. What are its essential features? Have you seen a problem like it before? What techniques are needed?

Try to make a rough estimate of the answer. It will help you understand the problem and will serve as a check against unreasonable answers. A car will not go 1,000 miles in 3 hours; a weight dropped from 10,000 feet will not hit the earth at 5 mph; the volume of a tank is not -275 gal.

Examine the data. Be sure you understand what is given. Translate the data into mathematical language. Whenever possible, make a clear diagram and label it accurately. Place axes to simplify computations. If you get stuck, check that you are using all the data.

Avoid sloppiness.

(a) *Avoid sloppiness in language.* Mathematics is written in English sentences. A typical mathematical sentence is "$y = 4x + 1$." The equal sign = is the verb in this sentence; it means *equals* or *is equal to*. The equal sign is not to be used in place of *and*, nor as a punctuation mark.

Quantities on opposite sides of an equal sign must be equal.

Use short simple sentences. Avoid pronouns such as "it" and "which". Give names and use them. Consider the following example.

"To find the minimum of it, differentiate it and set it equal to zero, then solve it which if you substitute it, it is the minimum."

Better: "To find the minimum of $f(x)$, set its derivative $f'(x)$ equal to zero. Let x_0 be the solution of the resulting equation. Then $f(x_0)$ is the minimum value of $f(x)$."

(b) *Avoid sloppiness in computation.* Do calculations in a sequence of neat, orderly steps. Include all steps except utterly trivial ones. This will help eliminate errors, or at least make errors easier to find. Check any numbers used; be sure that you have not dropped a minus sign or transposed digits.

(c) *Avoid sloppiness in units.* If you start out measuring in feet, all lengths must be in feet, all areas in square feet, and all volumes in cubic feet. Do not mix feet and acres, seconds and years.

(d) *Avoid sloppiness in the answer*. Be sure to answer the question that is asked. If the problem asks for the maximum value of $f(x)$, the answer is not the point where the maximum occurs. If the problem asks for a formula, the answer is not a number.

EXAMPLE Find the minimum of $f(x) = x^2 - 2x + 1$.

Solution 1:

$$2x - 2$$
$$x = 1$$
$$1^2 - 2 \cdot 1 + 1$$
$$0$$

Unbearable. This is just a collection of marks on the paper. There is absolutely no indication of what these marks mean or of what they have to do with the problem. When you write, it is your responsibility to inform readers what you are doing. Assume they are intelligent, but not mind readers.

Solution 2:

$$\frac{df}{dx} = 2x - 2 = 0 = 2x = 2 = x = 1$$
$$= f(x) = 1^2 - 2 \cdot 1 + 1 = 0.$$

Poor. The equal sign is badly mauled. This solution contains such enlightening statements as "0 = 2 = 1," and it does not explain what the writer is doing.

Solution 3:

$$\frac{df}{dx} = 2x - 2 = 0, \quad 2x = 2, \quad x = 1.$$

This is better than Solution 2, but contains two errors. Error 1: The first statement, "$\frac{df}{dx} = 2x - 2 = 0$," muddles two separate steps. First the derivative is computed, then the derivative is equated to zero. Error 2: The solution is incomplete because it does not give what the problem asks for, the minimum value of *f*. Instead, it gives the point *x* at which the minimum is assumed.

Solution 4: The derivative of *f* is

$$f' = 2x - 2.$$

At a minimum, **f '** = 0. Hence

$$2x - 2 = 0, \quad x = 1.$$

The corresponding value of *f* is

$$f(1) = 1^2 - 2 \cdot 1 + 1 = 0.$$

If $x > 1$, then $f'(x) = 2(x-1) > 0$, so *f* is increasing. If $x < 1$, then $f'(x) = 2(x-1) < 0$, so *f* is decreasing. Hence *f* is minimal at $x = 1$, and the minimum values of *f* is 0.

This solution is absolutely correct, but long. For homework assignments the following is satisfactory (check with your instructor):

Solution 5:

$$f'(x) = 2x - 2.$$

At min, $f' = 0$, $2x - 2 = 0$, $x = 1$. For $x > 1$, $f'(x) = 2(x-1) > 0$, *f*□: for $x < 1$, $f'(x) = 2(x-1) < 0$, *f*□.

Hence $x = 1$ yields min,

$$f_{min} = f(1) = 1^2 - 2 \cdot 1 + 1 = 0.$$

The next solution was submitted by a student who took a moment to think.

Solution 6:

$$f(x) = x^2 - 2x + 1 = (x-1)^2 \geq 0.$$

But

$$f(1) = (1-1)^2 = 0.$$

Hence the minimum value of $f(x)$ is 0.

- Adapated from *A First Course in Calculus*
  by Flanders, Korfhage and Price

MATH 223: Multivariable Calculus

Fall 2022

What To Do By Tomorrow

1) Read through the documents in this packet.
2) Access the course website http://s22.middlbury.edu/math0223a and check out some of the links.
3) Obtain a copy of our text. Read through the first chapter to refresh your memory of Single Variable Calculus and Linear Algebra. Examine the review sheets on vectors in the plane and 3-space.
4) Purchase a loose leaf binder to store the various handouts about the course that will be distributed, your class notes and the homework. You will accumulate a large number of loose sheets of paper from this course during the term; it's very helpful to keep them organized. You may also wish to obtain some graph paper.
5) Begin work on Assignment 1, the assignment due to be turned in on Wednesday.
6) **Don't hesitate to come in to see me if you need some help or have questions.**

VECTORS IN TWO DIMENSIONS

scalar quantity	scalar	directed line segment
initial point	terminal point	magnitude
equivalent vectors	equal vectors	velocity vector
force vector	displacement	sum of vectors
resultant force	scalar multiple	position vector
components of vector	$\mathbf{a} = (a_1, a_2)$	

Notation: $\mathbb{R}^2$ is the set of all vectors (x, y) where x and y are real numbers.

Definition: The **magnitude** $\|\mathbf{a}\|$ of the vector $\mathbf{a} = (a_1, a_2)$ is $\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2}$

Definition: (Addition of Vectors): $(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$

Definition: (Scalar Multiple of a Vector): $c\,(a_1, a_2) = (ca_1, ca_2)$

Definition: The **zero vector 0** is **O** = (0,0),
and the **negative of a** is $-\mathbf{a} = -(a_1, a_2) = (-a_1, -a_2)$

Theorem: The follow properties hold for vectors **a, b, w** and all scalars c and d:

$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$	$\mathbf{a} + (\mathbf{b} + \mathbf{w}) = (\mathbf{a} + \mathbf{b}) + \mathbf{w}$	$\mathbf{a} + \mathbf{0} = \mathbf{a}$
$\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$	$c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$	$(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a}$
$(cd)\mathbf{a} = c(d\mathbf{a}) = d(c\mathbf{a})$	$1\mathbf{a} = \mathbf{a}$	$0\mathbf{a} = \mathbf{0} = c\mathbf{0}$

Definition: If $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$, the **difference a - b** is **a + (-b).**

Theorem: If $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are any two points, then the vector **a** in $\mathbb{R}^2$ that corresponds to P_1P_2 is $\mathbf{a} = (x_2 - x_1, y_2 - y_1)$.

Definition: Nonzero vectors **a** and **b** in $\mathbb{R}^2$ have
(i) the **same direction** if $\mathbf{b} = c\mathbf{a}$ for some positive scalar $c > 0$.
(ii) the **opposite direction** if $\mathbf{b} = c\mathbf{a}$ for some negative scalar $c < 0$.
The vectors **a** and **b** are **parallel** if $\mathbf{b} = c\mathbf{a}$ for some scalar c

Theorem: If **a** is a vector and c is a scalar, then $\|c\,\mathbf{a}\| = |c|\,\|\mathbf{a}\|$

Definition: $\mathbf{i} = (1, 0)$, $\mathbf{j} = (0, 1)$

Theorem: If $\mathbf{a} = (a_1, a_2)$, then $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$

Definition: If $\mathbf{a} = (a_1, a_2)$, then a_1 is the **horizontal component** of **a** and a_2 is the **vertical component** of **a.**