Set Fundamentals

Sets

A ***set*** is a collection of objects, such that any object x is either in the set (written $x \in S$) or not in the set (written $x \notin S$), but not both: $\boxed{S \text{ is a set } \Leftrightarrow \forall x,\ x \in S \oplus x \notin S}$.

- Simple sets can be expressed simply by listing their elements — e.g., "the set $\{a, b, c\}$", or "the set $A = \{1, 2, 3, \ldots\}$".
- More complicated sets are often expressed via the notation $\{x : P(x)\}$, read "the set of all x such that $P(x)$."
  This allows us to collect all objects with some property into a set: $\boxed{a \in \{x : P(x)\} \text{ means "} P(a) \text{ is } \textit{true}\text{"}}$.
  - e.g., $a \in \left\{x : x^2 - 3x + 2 = 0\right\}$ simply means $a^2 - 3a + 2 = 0$.
- Some sets are expressed via the more intricate notation $\{f(x) : P(x)\}$, in which the left side isn't simply a variable.
  Here, the left side indicates the *form* of the set's elements, and the right side indicates the *credentials* required for inclusion into the set; in practical terms, $\boxed{a \in \{f(x) : P(x)\} \text{ means "} a = f(x)\text{, where } P(x) \text{ is } \textit{true}\text{"}}$.
  - e.g., $z \in \left\{x + iy : x, y \in \mathbb{R} \text{ and } x^2 + y^2 = 1\right\}$ means: $z = x + iy,$ where $x, y \in \mathbb{R}$ and $x^2 + y^2 = 1$.
- Some common sets and the symbols used to represent them:
  - The ***empty set***: $\varnothing = \{\ \}$, i.e., the set containing no elements
  - Some important sets of numbers:
    - the ***natural numbers***, $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$ and ***integers***, $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$;
    - the ***rationals***, $\mathbb{Q} = \left\{\frac{p}{q} : p, q \in \mathbb{Z} \text{ and } q \neq 0\right\}$ and ***real numbers***, $\mathbb{R}$; and
    - the ***complex numbers***, $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$ (where $i^2 = -1$).

Set Arithmetic

Numbers and logical propositions are not the only objects that can be manipulated and compared—similar operations exist for sets, which form the foundation of most objects that we in mathematics; the most important of these are listed below.

[In the formulæ below, uppercase letters represent sets and script letters represent collections of sets.]

Subsets	$A \subset B$ means $x \in A \Rightarrow x \in B$	
Equality	$A = B$ means $x \in A \Leftrightarrow x \in B$ [or, equivalently, $A \subset B \wedge B \subset A$]	
Union	$A \cup B \overset{\text{def}}{=} \{x : x \in A \vee x \in B\}$	$\bigcup \mathscr{B} \overset{\text{def}}{=} \{x : \exists B \in \mathscr{B} \text{ such that } x \in B\}$
Intersection	$A \cap B \overset{\text{def}}{=} \{x : x \in A \wedge x \in B\}$	$\bigcap \mathscr{B} \overset{\text{def}}{=} \{x : \forall B \in \mathscr{B},\ x \in B\}$
Difference	$A \setminus B \overset{\text{def}}{=} \{x : x \in A \wedge x \notin B\}$	$A \triangle B \overset{\text{def}}{=} \{x : x \in A \oplus x \in B\}$
Cartesian product	$A \times B \overset{\text{def}}{=} \{(a, b) : a \in A \wedge b \in B\}$	
Power set	$\mathscr{P}(X) \overset{\text{def}}{=} \{A : A \subset X\}$ [so $s \in \mathscr{P}(X) \Leftrightarrow s \subset X$]	
Distributive laws	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	$A \cap \bigcup_{B \in \mathscr{B}} B = \bigcup_{B \in \mathscr{B}} (A \cap B)$
	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	$A \cup \bigcap_{B \in \mathscr{B}} B = \bigcap_{B \in \mathscr{B}} (A \cup B)$
DeMorgan's laws	$X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)$	$X \setminus \bigcup_{B \in \mathscr{B}} B = \bigcap_{B \in \mathscr{B}} (X \setminus B)$
	$X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$	$X \setminus \bigcap_{B \in \mathscr{B}} B = \bigcup_{B \in \mathscr{B}} (X \setminus B)$
Emptiness	$A \neq \varnothing \Leftrightarrow \exists a \in A$	
	A and B are called ***disjoint*** if $A \cap B = \varnothing$	
	$\mathscr{C}$ is called a ***[pairwise] disjoint collection*** if $A, B \in \mathscr{C} \Rightarrow A = B$ or $A \cap B = \varnothing$	

Basic Set Theory & Logic

The concepts of set theory and logic are very closely connected, starting with their very basic principles:

- Given a set A and some object x, either $x \in A$ or $x \notin A$, but not both.
- A proposition P is either *true* or *false*, but not both.

In particular, from any set A, we obtain the statement that $x \in A$, connecting sets to statements; under this correspondence, each set concept directly relates to some logical concept, and [almost] vice-versa:

	Set Theory		Logic		Connection
Atomics					
	empty set	$\varnothing$	*false*		$x \in \varnothing \iff$ *false*
	universal set	$*$	*true*		Formally, no "universal set" exists
Unary operation					
	complement	$*$	$\neg$ / `!`	logical negation	The "complement" of $\varnothing$ would be universal
Binary operations					
	union	$\cup$	$\vee$ / `||`	or (inclusive!)	$x \in A \cup B \iff x \in A$ or $x \in B$
	intersection	$\cap$	$\wedge$ / `&&`	and	$x \in A \cap B \iff x \in A$ and $x \in B$
	set difference	$\setminus$	$\wedge\neg$ / `&&!`	and-not	$x \in A \setminus B \iff x \in A$ and $x \notin B$
	symmetric diff.	$\triangle$	$\oplus$ / `^`	exclusive-or	$x \in A \triangle B \iff x \in A \oplus x \in B$
Relations					
	subset	$\subset$	$\Rightarrow$	implies	$A \subset B$ means $x \in A \Rightarrow x \in B$
	equality	$=$	$\Leftrightarrow$	if and only if	$A = B$ means $x \in A \Leftrightarrow x \in B$
Operations on collections / Logical quantifiers					
	Union	$\bigcup$	$\exists$	existential quantifier	$x \in \bigcup \mathscr{A} \iff \exists A \in \mathscr{A} : x \in A$
	Intersection	$\bigcap$	$\forall$	universal quantifier	$x \in \bigcap \mathscr{A} \iff \forall A \in \mathscr{A}, x \in A$

- On the one hand, these concepts should reinforce one another due to shared meaning (and Venn diagrams!).
- On the other hand, each thing lives in just one world, that of sets or that of logic, so be careful to keep the symbols distinct in your head and your writing!