1. A RELATION ON A SET "A" IS A BINARY PREDICATE R THAT FOR EACH $a, b \in A$ ASSIGNS EITHER TRUE OR FALSE TO THE PROPOSITION "$a R b$".

(a) A RELATION CAN SIMPLY BE CONSIDERED TO BE A BINARY PREDICATE.

[THOUGH THE TRADITIONAL FORMAL DEFINITION OF A RELATION IN MATHEMATICS IS AS THE SET OF ORDERED PAIRS (a, b) FOR WHICH $a R b$ IS TRUE, WHICH IS NOT THE WAY YOU SHOULD CONCEPTUALIZE A RELATION.]

(b) SOME RELATIONS (PREDICATES!) WE'VE ALREADY SEEN ARE:

(i) ON $\mathbb{Z}$: $=$, (REFLEXIVE, SYMMETRIC, TRANSITIVE ∴ EQUIVALENCE RELATION!) $<, >$, (TRANSITIVE) $\leq, \geq$ (REFLEXIVE, TRANSITIVE)

(ii) ON THE SET OF VERTICES OF A GRAPH G:

- v_0 IS JOINED BY AN EDGE TO v_1 (SYMMETRIC)
- v_0 IS JOINED BY A PATH TO v_1 (REFLEXIVE, SYMMETRIC, TRANSITIVE ∴ EQUIVALENCE RELATION!)

(iii) ON THE SET OF SUBGRAPHS OF A GRAPH G:

- H IS A SUBGRAPH OF K (REFLEXIVE, TRANSITIVE)
- H IS A PROPER SUBGRAPH OF K (TRANSITIVE)
- H IS DISJOINT FROM K (SYMMETRIC)

2. A RELATION R ON A SET A IS...

(a) REFLEXIVE JUST WHEN $\forall a \in A,\ a R a$
(E.G., "IS JOINED BY A PATH TO", "IS A SUBGRAPH OF")

(b) SYMMETRIC JUST WHEN $\forall a, b \in A,\ a R b \Leftrightarrow b R a$
(E.G., "IS JOINED BY AN EDGE/PATH TO", "IS DISJOINT FROM")

(c) TRANSITIVE JUST WHEN $\forall a, b, c \in A,\ a R b \wedge b R c \Rightarrow a R c$.
(E.G., "IS JOINED BY A PATH TO", "IS A [PROPER] SUBGRAPH OF")

3. (a) AN EQUIVALENCE RELATION ON A SET A IS ONE THAT IS REFLEXIVE, SYMMETRIC, AND TRANSITIVE.

(b) FOR EACH $a \in A$, WE CAN FORM THE EQUIVALENCE CLASS OF a AS:

$$[a] = \{ b : a \, R \, b \}$$

ALSO SOMETIMES WRITTEN $\overline{a}$ OR $\mathbf{a}$

(c) THESE COLLECTION $\{ [a] : a \in A \}$ OF ALL EQUIVALENCE CLASSES FORM A PARTITION OF THE SET A BECAUSE

① EACH ONE IS NONEMPTY,
② DISTINCT EQUIVALENCE CLASSES ARE DISJOINT,
AND ③ THEIR UNION IS THE SET A.

(THEY BREAK THE SET A UP INTO PIECES, MUCH LIKE A JIGSAW PUZZLE)

IF OUR EQUIVALENCE RELATION IS "$\sim$", WE EXPRESS THIS COLLECTION OF EQUIVALENCE CLASSES BY "$A/\sim$".

(d) ANY GIVEN RELATION CAN BE "GROWN" INTO THE MINIMAL EQUIVALENCE RELATION SUBSUMING IT (I.E., SUCH THAT EVERY PAIR RELATED BY THE ORIGINAL RELATION IS STILL RELATED BY THE EQUIVALENCE RELATION).

FORMALLY, GIVEN ANY RELATION R ON A, WE CAN DEFINE SUCH AN EQUIVALENCE RELATION $\sim$ BY:

$$x \sim y \iff \forall \text{ EQUIVALENCE RELATIONS } S \text{ SUBSUMING } R, \; x \, S \, y.$$

4. WITH THE RELATION $\mid$ ON $\mathbb{Z}$ DEFINED BY $a \mid b \Leftrightarrow \exists k \in \mathbb{Z}$ WITH $b = ka$:
(SO $a \nmid b \Leftrightarrow \forall k \in \mathbb{Z},\ b \neq ka$)

USE CONTRADICTION! A "$\neq$" IS OFTEN MOST EASILY PROVEN THIS WAY, BECAUSE YOU GET TO USE "=" INSTEAD!

CLAIM: $4 \mid 12$, I.E., $\exists k \in \mathbb{Z}$ WITH $12 = k \cdot 4$

PROOF: TAKE $k = 3 \in \mathbb{Z}$. THEN $k \cdot 4 = 3 \cdot 4 = 12$, SO $4 \mid 12$ ∎

CLAIM: $6 \mid 12$, I.E., $\exists k \in \mathbb{Z}$ WITH $12 = k \cdot 2$

PROOF: TAKE $k = 2 \in \mathbb{Z}$. THEN $k \cdot 6 = 2 \cdot 6 = 12$, SO $6 \mid 12$ ∎

CLAIM: $12 \mid 12$, I.E., $\exists k \in \mathbb{Z}$ WITH $12 = k \cdot 12$

PROOF: TAKE $k = 1 \in \mathbb{Z}$. THEN $k \cdot 12 = 1 \cdot 12 = 12$, SO $12 \mid 12$ ∎

CLAIM: $24 \nmid 12$, I.E., $\forall k \in \mathbb{Z},\ 12 \neq k \cdot 24$

PROOF: LET $k \in \mathbb{Z}$ BE GIVEN, AND SUPPOSE (FOR CONTRADICTION) THAT $12 = k \cdot 24$.
THIS WOULD MEAN THAT $\frac{1}{2} = k$, CONTRADICTING $k \in \mathbb{Z}$. ∎

CLAIM: $5 \nmid 12$, I.E., $\forall k \in \mathbb{Z},\ 12 \neq k \cdot 5$

PROOF: LET $k \in \mathbb{Z}$ BE GIVEN, AND SUPPOSE (FOR CONTRADICTION) THAT $12 = k \cdot 5$.
THIS WOULD MEAN THAT $\frac{12}{5} = k$, CONTRADICTING $k \in \mathbb{Z}$. ∎

(b) THIS RELATION IS BOTH REFLEXIVE & TRANSITIVE. (TRY TO WRITE OUT & PROVE THESE!)

5. FIX A POSITIVE INTEGER n, AND DEFINE THE RELATION ON $\mathbb{Z}$:

$a \equiv b \pmod{n}$ JUST WHEN $n \mid (b-a)$.

(a) CLAIM: $6 \equiv 1 \pmod{5}$, I.E., $5 \mid (1-6)$, I.E. $5 \mid -5$, I.E. $\exists k \in \mathbb{Z}$ WITH $-5 = k \cdot 5$

PROOF: TAKE $k = -1 \in \mathbb{Z}$. THEN $k \cdot 5 = (-1) \cdot 5 = -5$, SO $5 \mid -5$,

AND THUS $6 \equiv 1 \pmod{5}$. ■

CLAIM: $4 \not\equiv 1 \pmod{5}$, I.E., $5 \nmid (1-4)$, I.E., $5 \nmid -3$, I.E., $\forall k \in \mathbb{Z}$, $-3 \neq k \cdot 5$

PROOF: LET $k \in \mathbb{Z}$ BE GIVEN, AND SUPPOSE (FOR CONTRADICTION) THAT $-3 = k \cdot 5$.

THIS WOULD MEAN THAT $-\frac{3}{5} = k$, CONTRADICTING $k \in \mathbb{Z}$,

SO $4 \not\equiv 1 \pmod{5}$. ■

(b) TO SHOW THAT $\equiv \pmod{n}$ IS AN EQUIVALENCE RELATION ON $\mathbb{Z}$, WE MUST SHOW REFLEXIVITY, SYMMETRY, AND TRANSITIVITY:

CLAIM: $\forall a \in \mathbb{Z}$, $a \equiv a \pmod{n}$ → I.E., $n \mid (a-a)$, OR $n \mid 0$,
I.E., $\exists k \in \mathbb{Z}$ WITH $0 = k \cdot n$

PROOF: LET $a \in \mathbb{Z}$ BE GIVEN.

TAKE $k = 0 \in \mathbb{Z}$. THEN $k \cdot n = 0 \cdot n = 0$, SO $n \mid 0$. ■

CLAIM: $\forall a, b \in \mathbb{Z}$, $a \equiv b \pmod{n} \Rightarrow b \equiv a \pmod{n}$ → $n \mid (a-b)$,
I.E., $\exists \ell \in \mathbb{Z}$ WITH $a - b = \ell \cdot n$

USE ANOTHER LETTER! ALREADY USED k

PROOF: LET $a, b \in \mathbb{Z}$ BE GIVEN,
AND SUPPOSE $a \equiv b \pmod{n}$, I.E., $n \mid (b-a)$, I.E., $\exists k \in \mathbb{Z}$ WITH $b - a = k \cdot n$

TAKING SUCH A $k \in \mathbb{Z}$, WE HAVE $b - a = kn$. $(*)$

TAKE $\ell = -k \in \mathbb{Z}$. THEN
$$\begin{aligned} \ell n &= (-k)n \\ &= -(kn) \\ &= -(b-a) \quad \text{BY } (*) \\ &= a-b \end{aligned}$$ ■

SCRATCH WORK: FIND $\ell \in \mathbb{Z}$ WITH $a - b = \ell n$
$b - a = kn$, SO NEGATING,
$a - b = -kn$ $\therefore \ell = -k$?

CLAIM: $\forall\, a,b,c \in \mathbb{Z}$, $a \equiv b \pmod n \wedge b \equiv c \pmod n \Rightarrow a \equiv c \pmod n$

i.e., $n \mid (c-a)$,
i.e., $\exists\, m \in \mathbb{Z}$ WITH $c-a = m \cdot n$

PROOF: LET $a,b,c \in \mathbb{Z}$ BE GIVEN, AND SUPPOSE:

- $a \equiv b \pmod n$, i.e., $n \mid (b-a)$, i.e. $\exists\, k \in \mathbb{Z}$ WITH $b-a = k \cdot n$

AND
- $b \equiv c \pmod n$, i.e., $n \mid (c-b)$, i.e., $\exists\, \ell \in \mathbb{Z}$ WITH $c-b = \ell \cdot n$

TAKING SUCH $k, \ell \in \mathbb{Z}$, WE HAVE $b-a = kn$ $(*)$
AND $c-b = \ell n$ $(**)$

SCRATCH WORK: FIND $m \in \mathbb{Z}$
WITH $c-a = mn$

$\left.\begin{array}{l} b-a = kn \\ c-b = \ell n \end{array}\right\}$ ADD: $c-a = (k+\ell)n$

$\therefore m = k+\ell$?

TAKE $m = k+\ell \in \mathbb{Z}$.

THEN
$$\begin{aligned} mn = (k+\ell)n &= kn + \ell n \\ &= (b-a) + (c-b) \\ &= \cancel{b} - a + c - \cancel{b} \\ &= c-a. \quad \blacksquare \end{aligned}$$

$\boxed{\text{EQUIVALENCE CLASSES}}$ FOR $n=1$ & $a \in \mathbb{Z}$, $\bar{a} = \{ b \in \mathbb{Z} : a \equiv b \pmod 1 \}$
$= \{ b \in \mathbb{Z} : 1 \mid (b-a) \}$

BUT $1 \mid$ EVERY INTEGER, SO $\forall\, b \in \mathbb{Z}$, $b \in \bar{a}$

$\therefore \bar{a} = \mathbb{Z}$

JUST ONE EQUIVALENCE CLASS, ALL OF $\mathbb{Z}$!

FOR $n=2$ & $a \in \mathbb{Z}$, $\bar{a} = \{ b \in \mathbb{Z} : a \equiv b \pmod 2 \}$
$= \{ b \in \mathbb{Z} : 2 \mid (b-a) \}$

IN OTHER WORDS, $b \in \bar{a}$ JUST WHEN $b-a$ IS EVEN,
OR $b = a +$ AN EVEN NUMBER, SO

$\bar{a} = \{ \dots, a-\ , a-4, a-2, a, a+2, a+4, a+6, \dots \}$

- IF a IS EVEN: $\{ \dots, -6, -4, -2, 0, 2, 4, 6, \dots \} = \bar{0}$
- IF a IS ODD: $\{ \dots, -5, -3, -1, 1, 3, 5, \dots \} = \bar{1}$

NOTE THAT EVERY INTEGER IS IN EXACTLY ONE OF THESE TWO SETS!

THESE ARE THE TWO EQUIVALENCE CLASSES

FOR $n=5$ & $a \in \mathbb{Z}$, THE EQUIVALENCE CLASSES SIMILARLY CONSIST OF INTEGERS DIFFERING BY MULTIPLES OF 5:

$$[a] = \{\ldots, a-10, a-5, a, a+5, a+10, \ldots\}$$

THERE ARE ONLY FIVE CASES, SO FIVE EQUIVALENCE CLASSES:

$$\{\ldots, -10, -5, 0, 5, 10, \ldots\} = \overline{0}$$
$$\{\ldots, -9, -4, 1, 6, 11, \ldots\} = \overline{1}$$
$$\{\ldots, -8, -3, 2, 7, 12, \ldots\} = \overline{2}$$
$$\{\ldots, -7, -2, 3, 8, 13, \ldots\} = \overline{3}$$

AND $\{\ldots, -6, -1, 4, 9, 14, \ldots\} = \overline{4}$

NOTE THAT EVERY INTEGER IS IN EXACTLY ONE OF THESE FIVE SETS!

IN GENERAL, THE EQUIVALENCE CLASSES MOD n WILL BE

$$\{\overline{0}, \overline{1}, \overline{2}, \ldots, \overline{n-1}\}$$

(DIVIDING ANY $a \in \mathbb{Z}$ BY n LEAVES EXACTLY ONE OF $0, 1, 2, \ldots, n-1$ AS THE REMAINDER, SO a AND THAT REMAINDER DIFFER BY A MULTIPLE OF n $\therefore$ ARE IN THE SAME EQUIVALENCE CLASS.)

6. FIX A POSITIVE INTEGER n, AND SUPPOSE $a, a', b, b' \in \mathbb{Z}$ SATISFY:

① $a \equiv a' \pmod{n}$, I.E., $n \mid (a'-a)$, I.E., $\exists\, k \in \mathbb{Z}$ WITH $a'-a = kn$ (*)

AND ② $b \equiv b' \pmod{n}$, I.E., $n \mid (b'-b)$, I.E. $\exists\, \ell \in \mathbb{Z}$ WITH $b'-b = \ell n$ (**)

(a) CLAIM: $a+b \equiv a'+b' \pmod{n}$, I.E., $n \mid [(a'+b')-(a+b)]$

PROOF: $(a'+b')-(a+b) = (a'-a) + (b'-b) = kn + \ell n = (k+\ell)n.$

(*) (**)

SINCE $k+\ell \in \mathbb{Z}$, $n \mid (k+\ell)n$, SO $n \mid [(a'+b')-(a+b)]$,

AND THUS $a+b \equiv a'+b' \pmod{n}$. ∎

(b) CLAIM: $a-b \equiv a'-b' \pmod{n}$, I.E., $n \mid [(a'-b')-(a-b)]$

PROOF: $(a'-b')-(a-b) = (a'-a) - (b'-b) = kn - \ell n = (k-\ell)n.$

(*) (**)

SINCE $k-\ell \in \mathbb{Z}$, $n \mid (k-\ell)n$, SO $n \mid [(a'-b')-(a-b)]$,

AND THUS $a-b \equiv a'-b' \pmod{n}$. ∎

(c) CLAIM: $a \cdot b \equiv a' \cdot b' \pmod{n}$, I.E., $n \mid (a' \cdot b' - a \cdot b)$

PROOF: $a' \cdot b' - a \cdot b = a' \cdot b' \underbrace{- a \cdot b' + a \cdot b'}_{} - a \cdot b$

↳ TRICK: ADD 0 TO GET $a'-a$ & $b'-b$ TO FACTOR OUT!

$= (a'-a)b' + a(b'-b)$

$= (kn)b' + a(\ell n)$ BY (*) & (**)

$= (kb' - a\ell)n.$

SINCE $kb' - a\ell \in \mathbb{Z}$, $n \mid (kb'-a\ell)n$, SO $n \mid (a' \cdot b' - a \cdot b)$,

AND THUS $a' \cdot b' \equiv a \cdot b \pmod{n}$. ∎

7. G: SIMPLE GRAPH, WITH TWO RELATIONS ON ITS SET OF VERTICES:

- $v_0 \; A \; v_1$ IF THERE IS AN EDGE JOINING v_0 TO v_1 (ADJACENT VERTICES)
- $v_0 \; C \; v_1$ IF THERE IS A PATH JOINING v_0 TO v_1 (VERTICES IN THE SAME COMPONENT)

C IS AN EQUIVALENCE RELATION (EASY TO CHECK REFLEXIVITY, SYMMETRY, & TRANSITIVITY), BUT A IS NOT (IT'S ONLY SYMMETRIC).

TWO VERTICES ARE EQUIVALENT UNDER C JUST WHEN THEY ARE CONNECTED BY A PATH, MEANING THAT THEY ARE IN THE SAME COMPONENT (WHY? THE PATH CONNECTING THEM IS NONEMPTY & CONNECTED, AND THUS IS CONTAINED IN SOME MAXIMAL NONEMPTY CONNECTED SUBGRAPH — I.E., COMPONENT — OF G.

$\therefore$ THE EQUIVALENCE CLASSES FOR C JUST CONSIST OF THE VERTICES IN EACH COMPONENT OF G

C IS THE EQUIVALENCE RELATION "GENERATED BY" A, BECAUSE TRANSITIVITY FORCES ALL VERTICES IN A CHAIN OF EDGES — I.E., A PATH — TO BE EQUIVALENT IN ANY EQUIVALENCE RELATION SUBSUMING A.

$v_0 \; A \; v_1 \; A \; v_2 \; A \cdots v_{n-1} \; A \; v_n \Rightarrow v_0 \equiv v_1$ IN THE EQUIVALENCE RELATION $\equiv$ (OR C) GENERATED BY A.

8. G: DIGRAPH; FOR VERTICES v_0, v_1 OF G,

- $v_0 \; SC \; v_1$ MEANS $\exists$ DIRECTED PATH (IN G) FROM v_0 TO v_1 AND $\exists$ DIRECTED PATH (IN G) FROM v_1 TO v_0

(a) TO SHOW THAT SC IS AN EQUIVALENCE RELATION ON THE VERTICES OF G, WE NEED TO SHOW REFLEXIVITY, SYMMETRY, AND TRANSITIVITY:

CLAIM: $\forall$ VERTICES a OF G, $a \; SC \; a$ — * NEED DIRECTED PATHS FROM v_0 TO v_1 & FROM v_1 TO v_0

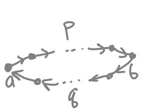
PROOF: LET A VERTEX a OF G BE GIVEN. $\bullet\, a$

THE CONSTANT PATH OF LENGTH 0 AT a CONNECTS a TO a. ■

CLAIM: $\forall$ VERTICES a, b OF G, $a \; SC \; b \Rightarrow b \; SC \; a$.

* NEED DIRECTED PATHS FROM b TO a & FROM a TO b

PROOF: LET VERTICES a, b OF G BE GIVEN,



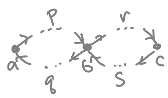
AND SUPPOSE THAT $a \; SC \; b$, I.E. $\exists$ DIRECTED PATHS p FROM a TO b & q FROM b TO a.

THEN q IS A DIRECTED PATH FROM b TO a, AND p IS A DIRECTED PATH FROM a TO b, SO BY DEFINITION, $b \; SC \; a$ ■

CLAIM: ∀ VERTICES a, b, c OF G, a sc b ∧ b sc c ⇒ a sc c.

*NEED DIRECTED PATHS FROM a TO c AND FROM c TO a

PROOF: LET VERTICES a, b, c OF G BE GIVEN,



AND SUPPOSE a sc b, I.E. ∃ DIRECTED PATHS p FROM a TO b & q FROM b TO a.

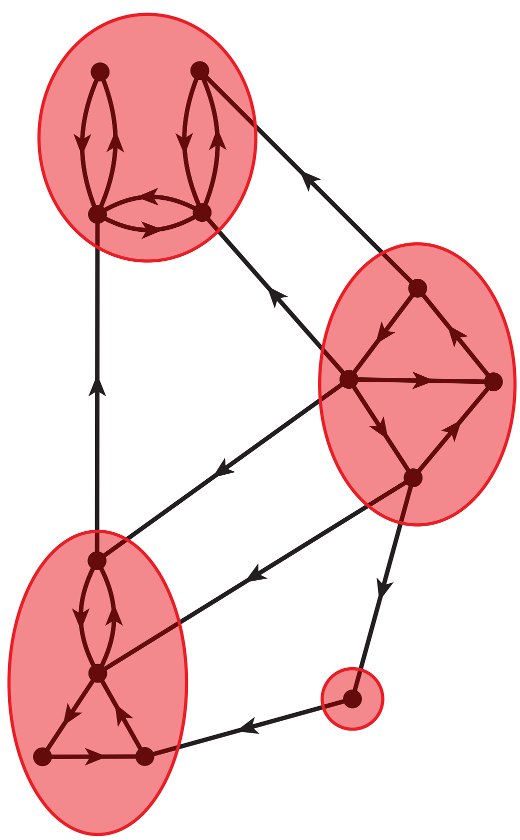
AND b sc c, I.E., ∃ DIRECTED PATHS r FROM b TO c & s FROM c TO b.

THEN CONCATENATING* THE PATHS p&r YIELDS A DIRECTED PATH FROM a TO c,
AND CONCATENATING* THE PATHS s&q YIELDS A DIRECTED PATH FROM c TO a,

SO BY DEFINITION, a sc c. ■

*TECHNICALLY, THE PATHS WE'RE CONCATENATING COULD SHARE SOME VERTICES OTHER THAN THEIR ENDPOINTS, BUT THIS CAN BE EASILY FIXED: LET v BE THE FIRST VERTEX ALONG p THAT IT SHARES WITH r, AND TRIM OFF EVERYTHING AFTER v IN p AND EVERYTHING BEFORE v IN r FIRST. (SIMILARLY FOR THE PATHS s&q.)

(b)

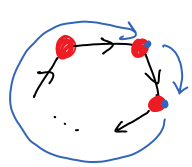


(c) (PICTURE EACH RED BLOB AS A BIG VERTEX TO SEE $\overline{G}$!)

$\overline{G}$ IS CLEARLY A DIRECTED GRAPH — WHY MUST IT BE ACYCLIC?
I.E., NOT CYCLIC.

PROVING ≠: CONTRADICTION??

SUPPOSE (FOR CONTRADICTION) THAT $\overline{G}$ HAD A DIRECTED CYCLE:



THIS WOULD GIVE DIRECTED PATHS IN BOTH DIRECTIONS BETWEEN VERTICES OF G IN DIFFERENT EQUIVALENCE CLASSES, MAKING THEM EQUIVALENT (CONTRADICTION!)