1. A STRING OVER THE ALPHABET Σ IS A FINITE SEQUENCE OF CHARACTERS FROM Σ.

(a) THESE ARE ALMOST EXACTLY LIKE STRINGS USED IN PROGRAMS, JUST USING CHARACTERS FROM A SPECIFIC GIVEN ALPHABET Σ.

E.G., IF $\Sigma = \{a,b,c\}$, abc AND $abbaccc$ ARE STRINGS OVER Σ

(b) ε STANDS FOR THE EMPTY STRING, WHICH CONTAINS NO CHARACTERS.

(c) $|x|$ DENOTES THE LENGTH OF THE STRING x, SO $|\varepsilon| = 0$.

2. THE PRODUCT xy OF TWO STRINGS x AND y IS SIMPLY THEIR...

NOTE THAT THIS IS NOT WRITTEN WITH +, AS IT IS IN CODE!

(a) CONCATENATION, I.E., THE SEQUENCE OF CHARACTERS FROM x FOLLOWED BY THOSE FROM y.

(b) THE STRING PRODUCT IS ASSOCIATIVE: $x(yz) = (xy)z$
BUT NOT COMMUTATIVE!

E.G., IF $x = abc$ AND $y = bba$, THEN $xy = abcbba$
AND $yx = bbaabc$

(ε ACTS AS ITS IDENTITY ELEMENT:
$x\varepsilon = x = \varepsilon x$ FOR EVERY STRING x)

(c) AS WITH NUMBERS, WE CAN DEFINE NONNEGATIVE WHOLE POWERS OF STRINGS INDUCTIVELY: $z^0 = \varepsilon$, AND $z^{n+1} = z^n \cdot z$ FOR $n \geq 0$.

$\therefore$ $z^0 = \varepsilon$, $z^1 = z$, $z^2 = zz$, $z^3 = zzz$, ETC.

(d) $\underbrace{|xy|}_{\text{LENGTH OF CONCATENATION}} = \underbrace{|x| + |y|}_{\text{SUM OF LENGTHS}}$ AND $\underbrace{|x^n|}_{\text{LENGTH OF A POWER OF } x} = \underbrace{n|x|}_{\text{NUMBER OF COPIES TIMES LENGTH OF } x}$

3. FOR A SET W OF STRINGS, WE CAN DEFINE ITS ASTERATE W^* INDUCTIVELY:

LET $\underbrace{W^0 = \{\varepsilon\}}_{\text{STRINGS BUILT BY CONCATENATING 0 STRINGS IN } W}$ AND $\underbrace{W^{n+1}}_{\text{STRINGS BUILT BY CONCATENATING (n+1) STRINGS IN } W} = \underbrace{W^n \cdot W}_{\text{ARE CONCATENATIONS OF STRINGS BUILT BY CONCATENATING } n \text{ STRINGS IN } W \text{ WITH ONE MORE STRING FROM } W}$ FOR $n \geq 0$; THEN $W^* = \bigcup_{n \geq 0} W^n$.

(!) W^* IS THE SET OF ALL STRINGS THAT CAN BE BUILT AS CONCATENATIONS OF SOME FINITE NUMBER OF STRINGS CHOSEN FROM W (WITH REPLACEMENT — STRINGS CAN BE USED MORE THAN ONCE!).

E.G., IF $W = \{ab, c, da\}$, THEN W^* CONTAINS ε, $abccda$, ccc, $cabdaabc$, ...

(a) A FEW KEY PROPERTIES ARE THAT $(W^*)^* = W^*$ AND $\varepsilon \in W^*$ FOR ANY SET W.

(b) IF x IS A SINGLE CHARACTER OR STRING, $x^* = \{x\}^* = \{\varepsilon, x, xx, xxx, xxxx, \ldots\}$
$= \{x^n : n \geq 0\}$

(c) Σ^* IS THE SET OF ALL [FINITE] STRINGS OVER Σ — SO $x \in \Sigma^*$ SIMPLY MEANS THAT x IS A STRING OVER Σ

4. IF X AND Y ARE STRINGS, THEIR FORMAL SUM $X+Y$ (NOT CONCATENATION!) REPRESENTS THE SET $\{X, Y\}$ OF BOTH STRINGS; WE CAN THINK OF IT AS COLLECTING A UNION OF SOME STRINGS OR SETS OF STRINGS.

THIS "+" HAS MOST OF THE PROPERTIES WE'RE USED TO FOR SUMS:

- IT'S ASSOCIATIVE: $(X+Y)+Z = X+(Y+Z)$ ← BOTH ARE $\{X, Y, Z\}$
- & COMMUTATIVE: $X+Y = Y+X$ ← BOTH ARE $\{X, Y\}$
- WITH $\emptyset$ AS ITS IDENTITY: $X+\emptyset = X = \emptyset + X$ ← ALL ARE $\{X\}$

IT ALSO HAS SOME PROPERTIES AS A UNION: $X+X=X$ ← BOTH JUST HAVE X

(!) PRODUCTS DISTRIBUTE OVER SUMS: $X(Y+Z) = XY + XZ$ ← CONCATENATING X AND (Y OR Z)

AND

$(X+Y)Z = XZ + YZ$ ← CONCATENATING (X OR Y) AND Z

5. A REGULAR EXPRESSION OVER Σ IS ANY EXPRESSION BUILT FROM:
- WORDS OVER Σ
- CONCATENATION $(\cdot)$
- SUMS $(+)$
- ASTERATES $(*)$
- AND $(\ )$ FOR GROUPING.

EACH REGULAR EXPRESSION GENERATES SOME SET OF STRINGS OVER Σ.

6.
- ε IS THE EMPTY STRING, AN ELEMENT OF Σ^*
- $\emptyset$ IS THE EMPTY SET, A SUBSET OF Σ^* CONTAINING NO ELEMENTS
- $\{\varepsilon\}$ IS A SET CONTAINING JUST THE EMPTY STRING, A SUBSET OF Σ^* WITH ONE ELEMENT, ε.

(BE CAREFUL, EACH OF THESE INVOLVES "EMPTY", BUT IN VERY DIFFERENT WAYS!)

7. $W^* \setminus \{\varepsilon\}$ IS EVERY STRING THAT CAN BE BUILT FROM THE STRINGS OF W, EXCEPT ε. IN OTHER WORDS, IT'S THE NONEMPTY STRINGS WE CAN BUILD FROM W.

IF $\varepsilon \notin W$, THIS IS DENOTED BY W^+ (TECHNICALLY, $W^+ = W \cdot W^*$ FOR ANY SET W; BUT IF $\varepsilon \in W$, THIS CONTAINS ε)

8. (a) $a^3\varepsilon^5(bca)^2 = \underline{aaabcabca}$

(b) $a(\varepsilon+b+c)w = (a+ab+ac)w = \underline{aw+abw+ac\,w}$

(c) $(\varepsilon+a)(\varepsilon+b) = (\varepsilon+a)\varepsilon + (\varepsilon+a)b$

$= \varepsilon\varepsilon + a\varepsilon + \varepsilon b + ab = \underline{\varepsilon + a + b + ab}$

(d) $(a+xy)^3 = (a+xy)^2(a+xy) = (a+xy)^2 a + (a+xy)^2 xy$

SIDE WORK: $(a+xy)^2 = (a+xy)(a+xy)$

$= (a+xy)a + (a+xy)xy$

$= a^2 + xya + axy + (xy)^2$

$= (a^2+xya+axy+(xy)^2)a + (a^2+xya+axy+(xy)^2)xy$

$= \underline{a^3 + xya^2 + axya + (xy)^2a + a^2xy + xyaxy + a(xy)^2 + (xy)^3}$

PHEW!

9. (a) 1^* GIVES ALL STRINGS OF 1'S, I.E., $\underline{\{1^n : n \geq 0\}}$

(b) 01^* GIVES THE STRINGS ABOVE WITH A PRECEDING 0, I.E., $\underline{\{01^n : n \geq 0\}}$

ASTERATES, LIKE POWERS, ARE EVALUATED BEFORE PRODUCTS, WHICH ARE EVALUATED BEFORE SUMS

(c) 0^*1 GIVES ALL STRINGS OF 0'S WITH A 1 AT THE END, I.E., $\underline{\{0^n1 : n \geq 0\}}$

(d) $(01)^*$ GIVES ALL POWERS OF 01, I.E., $\underline{\{(01)^n : n \geq 0\}}$

$\{\varepsilon, 01, 0101, 010101, \ldots\}$

(e) 0^*1^* GIVES THE CONCATENATIONS OF ALL STRINGS OF 0'S WITH ALL STRINGS OF 1'S,

I.E., $\{0^n1^m : n, m \geq 0\}$

NOTE THAT THE # OF 0'S NEED NOT MATCH THE # OF 1'S!

10. (a) THIS IS JUST THE CONCATENATION OF 101 WITH $\underbrace{\Sigma^* = (0+1)^*}_{\text{ANY STRING!}}$: $\underline{101(0+1)^*}$

(b) SIMILAR TO (a), BUT NOW WE COULD HAVE ANY STRING AT THE START, AS WELL:

$$\underline{(0+1)^*000(0+1)^*}$$

(c) THIS IS JUST THE ASTERATE OF 001: $\underline{(001)^*}$

(d) A LITTLE TRICKY!

THIS MEANS THAT WE HAVE ALL OF OUR 1'S AT THE END (ANY # OF THEM, SO 1^*), AND THUS ALL 0'S AT THE START (ANY # OF THEM, SO 0^*).

CONCATENATING, WE GET $\underline{0^*1^*}$