1. (a) A COLUMN VECTOR OF DIMENSION k SIMPLY CONSISTS OF (k) REAL NUMBERS ARRANGED INTO A COLUMN.

$$\underbrace{\begin{bmatrix} 1 \\ 5 \\ -1 \\ 2 \end{bmatrix}}_{\text{COLUMN VECTOR OF DIMENSION 4}} \qquad \underbrace{\begin{bmatrix} 3 & 2 \\ 0 & -4 \\ 1 & -3 \end{bmatrix}}_{3\times 2 \text{ MATRIX}}$$

AN $m \times n$ MATRIX IS A RECTANGULAR ARRANGEMENT OF REAL NUMBERS, WITH (m) ROWS AND (n) COLUMNS.

* AN $m \times n$ MATRIX IS BEST CONSIDERED AS BEING BUILT FROM (n) m-DIMENSIONAL COLUMN VECTORS.

(b) IF A IS AN $m \times n$ MATRIX AND $\vec{x}$ IS AN n-DIMENSIONAL COLUMN VECTOR, $A\vec{x}$ IS THE LINEAR COMBINATION OF THE COLUMNS OF A WITH THE COEFFICIENT FOR EACH COLUMN GIVEN BY THE ENTRIES OF $\vec{x}$, RESULTING IN SOME m-DIMENSIONAL COLUMN VECTOR.

IN THIS WAY, THE MATRIX A REPRESENTS A FUNCTION MAPPING n-DIMENSIONAL COLUMN VECTORS TO m-DIMENSIONAL COLUMN VECTORS:

$$A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

ITS DOMAIN IS $\mathbb{R}^n$ (THE SET OF ALL n-DIMENSIONAL COLUMN VECTORS) AND ITS CODOMAIN IS $\mathbb{R}^m$ (... m-DIMENSIONAL...).

E.G.:

$$\underbrace{\begin{bmatrix} 1 & 5 & 2 & 3 \\ 2 & 0 & 2 & 0 \\ 3 & -1 & 1 & -3 \end{bmatrix}}_{3 \times 4 \text{ MATRIX}} \underbrace{\begin{bmatrix} 1 \\ 0 \\ -2 \\ 3 \end{bmatrix}}_{\text{COLUMN VECTOR IN } \mathbb{R}^4} = 1\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 0\begin{bmatrix} 5 \\ 0 \\ -1 \end{bmatrix} + (-2)\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -4 \\ -4 \\ -2 \end{bmatrix} + \begin{bmatrix} 9 \\ 0 \\ -9 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 6 \\ -2 \\ -8 \end{bmatrix}}_{\text{COLUMN VECTOR IN } \mathbb{R}^3}$$

(c) FOR CONVENIENCE, LET'S WRITE $\vec{e}_j$ FOR A COLUMN VECTOR THAT'S ALL ZEROES EXCEPT FOR A 1 IN THE j^{TH} SPOT.

THEN FROM (6), $\boxed{A\vec{e}_j = \text{THE } j^{\text{TH}} \text{ COLUMN OF } A}$ (WHY?)

THUS, IF A IS AN $m \times n$ MATRIX AND B IS AN $n \times p$ MATRIX, THEN:

$$(AB)\vec{e}_j = j^{\text{TH}} \text{ COLUMN OF } AB$$

ON THE OTHER HAND, $(AB)\vec{e}_j = A(B\vec{e}_j)$

$$= A(j^{\text{TH}} \text{ COLUMN OF } B)$$

SO: $\boxed{\text{THE } j^{\text{TH}} \text{ COLUMN OF } AB = A(j^{\text{TH}} \text{ COLUMN OF } B)}$

I.E., THE LINEAR COMBINATION OF THE COLUMNS OF A WITH COEFFICIENTS GIVEN BY THE j^{TH} COLUMN OF B.

KEEP IN MIND:
- AB IS A <u>COMPOSITION OF FUNCTIONS</u>, SO
- THE OPERATIONS ACT <u>RIGHT-TO-LEFT</u>, AND
- <u>ORDER MATTERS</u>!!

2. IF $A = \begin{bmatrix} a & b & x_0 \\ c & d & y_0 \\ 0 & 0 & 1 \end{bmatrix}$, THEN

(a) $A(0,0) = \begin{bmatrix} a & b & x_0 \\ c & d & y_0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ 1 \end{bmatrix} = (x_0, y_0)$

(b) $A\langle 1,0\rangle = \begin{bmatrix} a & b & x_0 \\ c & d & y_0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \\ 0 \end{bmatrix} = \langle a, c\rangle$, AND

$A\langle 0,1\rangle = \begin{bmatrix} a & b & x_0 \\ c & d & y_0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} b \\ d \\ 0 \end{bmatrix} = \langle b, d\rangle$.

3. AS WE'LL SEE IN PROBLEM #4, OUR "STANDARD" ROTATION & SCALING MATRICES ARE CENTERED AT THE ORIGIN (0,0), WHICH DOESN'T MOVE.

IF WE WANT TO ROTATE ABOUT OR SCALE FROM SOME OTHER FIXED POINT (x_0, y_0), WE CAN DO THIS BY <u>CONJUGATING</u> THE STANDARD TRANSFORMATION BY THE APPROPRIATE TRANSLATION:

(REMEMBER TO COMPOSE THESE RIGHT-TO-LEFT!)

① TRANSLATE BY $\langle -x_0, -y_0 \rangle$,

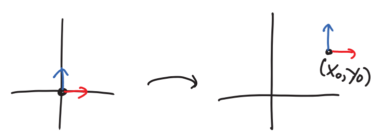
② (PERFORM A TRANSFORMATION ABOUT (0,0))

③ TRANSLATE BY $\langle x_0, y_0 \rangle$

INVERSES OF ONE ANOTHER — SANDWICHING AN OPERATION BETWEEN SOME OTHER OPERATION AND ITS INVERSE IS CALLED <u>CONJUGATION</u>

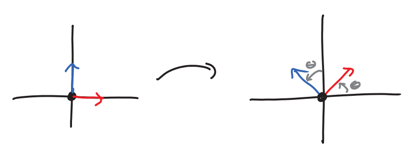
4. (TO FIND WHERE EACH OF THESE MAPS (0,0), $\langle 1,0 \rangle$, AND $\langle 0,1 \rangle$, USE AFFINE COORDINATES AS IN PROBLEM #2.)

$T_{\langle x_0, y_0 \rangle}$ MAPS $\begin{cases} (0,0) \mapsto (x_0, y_0) \\ \langle 1,0 \rangle \mapsto \langle 1,0 \rangle \\ \langle 0,1 \rangle \mapsto \langle 0,1 \rangle \end{cases}$



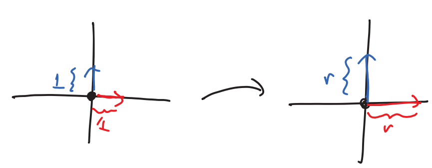
TRANSLATION BY $\langle x_0, y_0 \rangle$

R_θ MAPS $\begin{cases} (0,0) \mapsto (0,0) \\ \langle 1,0 \rangle \mapsto \langle \cos\theta, \sin\theta \rangle \\ \langle 0,1 \rangle \mapsto \langle -\sin\theta, \cos\theta \rangle \end{cases}$



ROTATION ↺ BY θ ABOUT (0,0)

S_r MAPS $\begin{cases} (0,0) \mapsto (0,0) \\ \langle 1,0 \rangle \mapsto \langle r,0 \rangle \\ \langle 0,1 \rangle \mapsto \langle 0,r \rangle \end{cases}$



SCALE BY FACTOR OF r FROM (0,0)

5. * DON'T FORGET TO COMPOSE RIGHT-TO-LEFT!

(a) $T_{\langle 1,1\rangle} S_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}}}$

* WHEN TRANSLATION IS DONE LAST, WE CAN JUST ADD IT TO THE IN THE TOP-RIGHT VECTOR

(b) $S_2 T_{\langle 1,1\rangle} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}}}$

** WHEN IT'S NOT DONE LAST, THE OTHER TRANSFORMATIONS ACT ON THE TRANSLATION VECTOR

(c) $T_{\langle 0,2\rangle} R_{\pi/2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ *

(d) $R_{\pi/2} T_{\langle 0,2\rangle} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **

(e) $S_2 R_{\pi/2} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

SAME!

(f) $R_{\pi/2} S_2 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

THE T'S COMMUTE WITH EACH OTHER; FROM (e,f), WE MIGHT GUESS THAT THE R_θ'S AND S_r'S ALL COMMUTE WITH EACH OTHER, AND WE COULD PROVE THIS BY JUST COMPUTING $S_r R_\theta$ AND $R_\theta S_r$ IN GENERAL:

$$S_r R_\theta = \begin{bmatrix} r\cos\theta & -r\sin\theta & 0 \\ r\sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = R_\theta S_r$$

BUT TRANSLATIONS, IN GENERAL, DO NOT COMMUTE WITH ROTATIONS & SCALING.

6. (a) TO ROTATE ABOUT $(1,3)$: TRANSLATE BY $\langle -1,-3\rangle$, ROTATE, THEN TRANSLATE BY $\langle 1,3\rangle$:

COUNTERCLOCKWISE IS NEGATIVE!

$$T_{\langle 1,3\rangle} R_{(-\pi/4)} T_{\langle -1,-3\rangle} = \underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & \frac{1}{\sqrt{2}} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

NOTE $\cos(-\pi/4) = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ $\left(\text{OR } \frac{\sqrt{2}}{2}\right)$

& $\sin(-\pi/4) = -\sin\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$ $\left(\text{OR } -\frac{\sqrt{2}}{2}\right)$

$$\overset{*}{=} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 1 \\ -1/\sqrt{2} & 1/\sqrt{2} & 3 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & (1 - 1/\sqrt{2} - 3/\sqrt{2}) \\ -1/\sqrt{2} & 1/\sqrt{2} & (3 + 1/\sqrt{2} - 3/\sqrt{2}) \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \underline{\underline{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & (1-2\sqrt{2}) \\ -1/\sqrt{2} & 1/\sqrt{2} & (3-\sqrt{2}) \\ 0 & 0 & 1 \end{bmatrix}}}$$

CHECKING THAT THIS MATRIX FIXES $(1,3)$:

$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & (1-2\sqrt{2}) \\ -1/\sqrt{2} & 1/\sqrt{2} & (3-\sqrt{2}) \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = 1\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix} + 3\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} + 1\begin{bmatrix} 1-2\sqrt{2} \\ 3-\sqrt{2} \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cancel{1/\sqrt{2}} + \cancel{3/\sqrt{2}} + 1 - \cancel{2\sqrt{2}} \\ -\cancel{1/\sqrt{2}} + \cancel{3/\sqrt{2}} + 3 - \cancel{\sqrt{2}} \\ 0+0+1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \checkmark$$

(b) TO SCALE FROM $(2,4)$: TRANSLATE BY $\langle -2,-4\rangle$, THEN SCALE, THEN TRANSLATE BY $\langle 2,4\rangle$:

$$T_{\langle 2,4\rangle} S_5 T_{\langle -2,-4\rangle} = \underbrace{\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}}\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\overset{*}{=} \begin{bmatrix} 5 & 0 & 2 \\ 0 & 5 & 4 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 5 & 0 & -8 \\ 0 & 5 & -16 \\ 0 & 0 & 1 \end{bmatrix}}}$$

CHECKING THAT THIS MATRIX FIXES $(2,4)$:

$$\begin{bmatrix} 5 & 0 & -8 \\ 0 & 5 & -16 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix} + 1\begin{bmatrix} -8 \\ -16 \\ 1 \end{bmatrix} = \begin{bmatrix} 10+0-8 \\ 0+20-16 \\ 0+0+1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} \checkmark$$

7. $$T_{\langle 4,0\rangle} R_{\pi/2} T_{\langle -1,-1\rangle} S_2 = \underbrace{\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}\underbrace{\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$$\overset{*,*}{=} \begin{bmatrix} 0 & -1 & 4 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 0 & -2 & 5 \\ 2 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}}}$$