Relations and Functions: definitions

1. If $A$ and $B$ are two sets, what is a relation $R$ from $A$ to $B$, and how does this differ from the single-set case? What are the domain and codomain of such a relation $R$?

2. What is the defining property of a function $f : A \to B$ (best written as a relation $f$)? How should we conceptualize what a function does? If $a \in A$, what do we mean by the expression $f(a)$ (what is less than ideal about the phrase “the function $f(x)$”)?

3. What does it mean for two functions $f, f' : A \to B$ to be equal?

4. If $f : A \to B$ is a function, discuss each of the following terms, both intuititively and symbolically:
   (a) $f$ is injective (or one-to-one)—how does this relate to the definition of $f$ being a function?
   (b) $f$ is surjective (or onto)—how does this relate to the range of $f$?
   (c) $f$ is bijective (or one-to-one & onto)—what is special about bijective functions?

5. Given functions $f : A \to B$ and $g : B \to C$, define their composition $g \circ f$—what are this function’s domain and codomain, and how is the function defined? [Be very careful to note that the functions of a composition are read right-to-left!]

6. For any set $X$, how do we define the identity function? Which of the above function properties does it possess? What happens when an identity function appears in a composition?

7. What does it mean for two functions $f : A \to B$ and $g : B \to A$ to be inverses of one another (formally, intuitively, and symbolically)? How does this relate to identity functions?

...and key results and applications

8. Supposing $f : A \to B$ and $g : B \to C$, prove each of the following propositions (the proofs will easily click into place if you’re careful with your new definitions, use of variables, and basic proof techniques!):
   (a) If $f$ and $g$ are both injective, then $g \circ f$ is injective.
   (b) If $g \circ f$ is injective, then $f$ is injective.
   (c) If $f$ and $g$ are both surjective, then $g \circ f$ is surjective.
   (d) If $g \circ f$ is surjective, then $g$ is surjective.

9. Using the function formulation of inverses, what do 8(b) and 8(d) tell us if $f : A \to B$ and $g : B \to A$ are inverses?

10. While it might seem “obvious*”, each bijective function $f : A \to B$ has just one inverse. Use the definition of inverse functions to show that if $g, g' : B \to A$ are both inverses for $f$, then $g = g'$.
     [Hint: Consider the composition $g \circ f \circ g'$ grouped with parentheses in two different ways.]
     How do we denote this unique inverse of a bijective function $f$?

11. Suppose that $f : A \to B$ and $g : B \to C$ are both invertible. How do we know that $g \circ f$ is invertible, and what is the formula for its inverse (be careful with order!)?

12. Consider the function mapping graphs to $\mathbb{Z}$ via $G \mapsto$ number of vertices in $G$.
    (a) Explain why it is that if $G$ and $H$ are isomorphic graphs, then $v(G) = v(H)$.
    (b) What does this tell us if two graphs have different numbers of vertices?
    (c) What does this tell us if two graphs have the same number of vertices?
    (d) In the above contexts, we are using $f$ as an invariant of isomorphism-classes of graphs (isomorphism is an equivalence relation!). Try to construct some other invariants of isomorphism classes of graphs.