Relations and Functions: definitions

1. If A and B are *two* sets, what is a ***relation*** R *from* A *to* B, and how does this differ from the single-set case? What are the ***domain*** and ***codomain*** of such a relation R?

2. What is the defining property of a ***function*** $f : A \to B$ (best written as a relation $\overset{f}{\mapsto}$)?
   How should we conceptualize what a function *does*?
   If $a \in A$, what do we mean by the expression $f(a)$ (what is less than ideal about the phrase "the function $f(x)$")?

3. What does it mean for two functions $f, f' : A \to B$ to be equal?

4. If $f : A \to B$ is a function, discuss each of the following terms, both *intuititively* and *symbolically*:
   (a) f is ***injective*** (or ***one-to-one***)—how does this relate to the definition of f being a function?
   (b) f is ***surjective*** (or ***onto***)—how does this relate to the ***range*** of f?
   (c) f is ***bijective*** (or ***one-to-one & onto***)—what is special about bijective functions?

5. Given functions $f : A \to B$ and $g : B \to C$, define their ***composition*** $g \circ f$—what are this function's domain and codomain, and how is the function defined?
   [Be very careful to note that the functions of a composition are read right-to-left!]

6. For any set X, how do we define the ***identity function***? Which of the above function properties does it possess? What happens when an identity function appears in a composition?

7. What does it mean for two functions $f : A \to B$ and $g : B \to A$ to be ***inverses*** of one another (formally, intutively, and symbolically)? How does this relate to identity functions?

...and key results and applications

8. Supposing $f : A \to B$ and $g : B \to C$, prove each of the following propositions (the proofs will easily click into place if you're careful with your new definitions, use of variables, and basic proof techniques!):
   (a) If f and g are both injective, then $g \circ f$ is injective.
   (b) If $g \circ f$ is injective, then f is injective.
   (c) If f and g are both surjective, then $g \circ f$ is surjective.
   (d) If $g \circ f$ is surjective, then g is surjective.

9. Using the function formulation of inverses, what do 8(b) and 8(d) tell us if $f : A \to B$ and $g : B \to A$ are inverses?

10. While it might seem "obvious*", each bijective function $f : A \to B$ has just one inverse.
    Use the definition of inverse functions to show that if $g, g' : B \to A$ are both inverses for f, then $g = g'$.
    [Hint: Consider the composition $g \circ f \circ g'$ grouped with parentheses in two different ways.]
    How do we denote this unique inverse of a bijective function f?

11. Suppose that $f : A \to B$ and $g : B \to C$ are both invertible. How do we know that $g \circ f$ is invertible, and what is the formula for its inverse (be careful with order!)?

12. Consider the function mapping graphs to $\mathbb{Z}$ via $G \overset{v}{\mapsto}$ number of vertices in G.
    (a) Explain why it is that if G and H are isomorphic graphs, then $v(G) = v(H)$.
    (b) What does this tell us if two graphs have *different* numbers of vertices?
    (c) What does this tell us if two graphs have the *same* number of vertices?
    (d) In the above contexts, we are using f as an ***invariant*** of isomorphism-classes of graphs (isomorphism is an equivalence relation!). Try to construct some other invariants of isomorphism classes of graphs.