Graphs: Cycles and Trees

0. Keep playing with the online exercises to solidify the basic concepts and terminology! Your intuition for these concepts will be built largely visually, and that be essential in helping to you work through formal details without getting lost.

> I've transitioned to using words in the propositions below: *if-then*, *every*, *there is*, etc. Practice reading these as the appropriate logical constructions $\Rightarrow$, $\forall$, and $\exists$, as you should already know exactly how direct proofs of such statements proceed!
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> Recall that the ***contrapositive*** of $P \Rightarrow Q$ is $\neg Q \Rightarrow \neg P$ (these two propositions are logically equivalent), and that we can also prove $P \Rightarrow Q$ by ***contradiction***, showing that $\neg[P \Rightarrow Q]$—i.e., that $P \wedge \neg Q$—is *false*.

1. Prove the following basic results about cycles and acyclic graphs:
   (a) Prove that for any graph G, if G has a loop or a parallel edge, then it is *not* acyclic.
   (b) Change the blue bit above to its contrapositive—what does this tell us?
   (c) What is the minimum number of vertices in a *simple* cycle? Why?
   (d) Prove that for every simple cycle C, there exist vertices v_0, v_1 of C with two distinct paths connecting them (a picture will suffice).

2. Prove the following fundamental results regarding trees, using induction on the number $n \geq 1$ of vertices:
   (a) Every tree T is a simple graph.
   (**b) For every tree T and each pair of vertices v_0, v_1 of T, there is a *unique* path in T connecting v_0 and v_1.
   (c) If T is a tree, then $n(T) = e(T) + 1$.
   $n(T)$ and $e(T)$ represent the numbers of vertices and edges of T, respectively.

3. Deduce the following results about trees:
   (a) Every tree T is nonempty. Use the definition of a tree.
   (b) Every tree T is acyclic.
   Strategy: see 2(b) & 1(d)—suppose that T contains a cycle C to obtain a ***contradiction***!
   (c) Every tree T is connected. See 2(b).
   (d) If a graph G has a spanning tree T, then G is connected. See previous part.

4. Prove that if G is nonempty and connected, and if G is *not* a tree, then G contains a cycle.

   Strategy: start by taking a spanning tree T for G,
   and deduce that G has an edge not in T;
   use 2(b) and this edge to form a cycle in G.

   Change the blue bit to its contrapositive; how does the resulting statement relate to problem 3(a,b,c)?

5. Suppose that we number the vertices of a simple graph G as $1, 2, 3, \ldots n$, and that we have a list of pairs of integers $\{ v_0, v_1 \}$ representing the edges of G. Outline algorithms for doing each of the following:
   (a) Finding the components of a graph G.
   (b) Finding a spanning tree for a connected graph G.
   (c) Finding a spanning forest for a [possibly disconnected] graph G.