Proof by Induction

The principle of mathematical induction: to prove a proposition of the form $\forall n \geq 0,\ P(n)$, we can instead prove: ① $P(0)$

and ② $\forall n \geq 0,\ P(n) \Rightarrow P(n+1)$.

Take care substituting $n \rightsquigarrow (n+1)$

1. Use induction on n to prove that $\forall n \geq 0,\ 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}(2n^3 + 3n^2 + n)$.

2. Suppose that $r \neq 1$; use induction on n to prove that $\forall n \geq 0,\ 1 + r + r^2 + \cdots + r^n = \dfrac{1 - r^{n+1}}{1 - r}$.

3. Use induction on n to prove the following inequalities:

   (a) $\forall n \geq 10,\ 100n < 2^n$

   (b) $\forall n \geq 4,\ 2^n < n!$ — Recall that $n! = 1 \cdot 2 \cdot 3 \cdots n$

   (c) $\forall n \geq 0,\ (1+x)^n \geq 1 + nx$ — Suppose that $x > -1$ for this part!

. . . and the Fibonacci sequence

The *Fibonacci sequence* $F_0, F_1, F_2, F_3, \ldots$ is defined by $F_0 = 0$,

$F_1 = 1$, and

$$\forall n \geq 1,\ \boxed{F_{n+1} = F_n + F_{n-1}}.$$

4. Use the definition above to compute the first eleven terms of the Fibonacci sequence, $F_0, F_1, \ldots, F_{10}$.

5. Prove the following Fibonacci identities using induction on n:

   (a) $\forall n \geq 0,\ F_0 + F_1 + F_2 + \cdots + F_n = F_{n+2} - 1$

   (b) $\forall n \geq 0,\ F_0^2 + F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1}$

   (c) $\forall n \geq 0,\ F_{2n+1} = F_{n+1}^2 + F_n^2 \ \wedge\ F_{2n+2} = F_{n+2}^2 - F_n^2$

   (*d) $\forall n \geq 1,\ F_{n+1} F_{n-1} = F_n^2 + (-1)^n$

6. Note that the results of problem 2(c) allow us to compute later Fibonacci numbers (with both odd and even indices) in terms of *much* earlier ones.

   (a) Use your answers to problem 1 to double-check what they say about F_9 and F_{10}.

   Feel free to use a calculator or computer to help with the arithmetic in parts (b,c,d)!

   (b) Use them to find the values of F_{15}, F_{16}, and F_{17}.

   (c) Now use them again to find the values of F_{31}, F_{32}, and F_{33}.

   (d) How do your three values in part (b) relate to one another?
   How about those in part (c)?