Direct Proofs

Define the following subsets of $\mathbb{Z}$:

$E = \{\, x \in \mathbb{Z} \,:\, \exists\, k \in \mathbb{Z} \text{ such that } x = 2k \,\}$ *even*

$O = \{\, x \in \mathbb{Z} \,:\, \exists\, k \in \mathbb{Z} \text{ such that } x = 2k+1 \,\}$ *odd*

$T = \{\, 3k \,:\, k \in \mathbb{Z} \,\}$ *threven*

$U = \{\, 6k+1 \,:\, k \in \mathbb{Z} \,\}$ *unty*

$Q = \{\, 6k+5 \,:\, k \in \mathbb{Z} \,\}$ *quinty*

Remember that the variables used in the set notations above are not significant—don't ever use the same variable to represent more than one thing in a given context!

Prove the following propositions:

1. $x, y \in O \;\Rightarrow\; x+y \in E.$
2. $x \in E \wedge y \in O \;\Rightarrow\; xy \in E.$
3. $x \in O \;\Rightarrow\; \frac{x+1}{2} \in \mathbb{Z}.$
4. $\mathbb{Z} = E \cup O.$ [Hint: Use (*) with $b = 2$ and split cases.]
5. $U \subset O$
6. $x \in T \;\Rightarrow\; 2x - 5 \in U$
7. $x \in Q \;\Rightarrow\; x^2 \in U$
8. $x \in Q \wedge y \in U \;\Rightarrow\; x - y \in E$
9. $\mathbb{Z} = E \cup T \cup U \cup Q$ [Hint: Use (*) with $b = 6$ and split cases.]
10. $x \in Q \;\wedge\; y \in U \;\Rightarrow\; x + y \in E \cap T$

* Keep in mind the *division algorithm:* Given any $a, b \in \mathbb{Z}$ with $b > 0$, we can write $a = bq + r$ for some $q, r \in \mathbb{Z}$ with $0 \le r < b$.

$\frac{a}{b} = q + \frac{r}{b}$; q is the quotient and r is the remainder.