String algebra: in principle

Suppose that $\Sigma$ is a finite “alphabet” (set) of characters or symbols.

1. Define what we mean by a string (or word) over the alphabet $\Sigma$.
   (a) Compare the abstract concept of a string with those used in programs.
   (b) What do we mean by the string $\varepsilon$?
   (c) Given a string $x$, what does $|x|$ represent? What is $|\varepsilon|$?

2. If $x$ and $y$ are strings over $\Sigma$, define their product $xy$, i.e., $x \cdot y$.
   (a) What other term describes this operation?
   (b) What properties does this operation have, and—crucially—what key property does it not have?
   (c) How does the product let us define nonnegative whole powers $z^n$ of a string $z$?
   (d) If $x$ and $y$ are strings, how can we simplify $|xy|$ and $|x^n|$?

3. If $W$ is a set of strings, define its asterate $W^*$.
   (a) What properties does this operation have, and what string does this set always include?
   (b) If $x$ is a single character or string over $\Sigma$, we can define $x^* = \{x\}^*$.
      What strings does the set $x^*$ contain?
   (c) In words, what is $\Sigma^*$, and what does the statement “$x \in \Sigma^*$” mean?

4. Blurring the line between strings and sets of strings a bit, if $x$ and $y$ are strings, what is meant by their formal sum $x + y$, and what set operation does this correspond to?
   What properties does this operation have, and how does it relate to our other operation “·”?

5. What is a regular expression over $\Sigma$?

...and in practice

In the problems below, characters in $\Sigma$ will be written in typewriter font, with variables representing strings written as usual via italic letters.

6. Constrast, in writing, the meanings of the following, including the context of whether each is an element of $\Sigma^*$ or a subset of $\Sigma^*$:
   (a) $\varepsilon$  (b) $\emptyset$  (c) $\{\varepsilon\}$

7. For any set $W$ of strings, describe in writing what $W^* \setminus \{\varepsilon\}$ means.
   [This comes up enough that we often denote it by $W^+$.]

8. Simplify the following string expressions:
   (a) $a^3\varepsilon^5(bca)^2$  (b) $a(\varepsilon + b + c)w$  (c) $(\varepsilon + a)(\varepsilon + b)$  (d) $(a + xy)^3$

9. Describe the sets of strings generated by the following regular expressions over $\Sigma = \{0, 1\}$, both in writing and using set-builder notation:
   (a) $1^*$  (b) $01^*$  (c) $0^*1$  (d) $(01)^*$  (e) $0^*1^*$

10. Find regular expressions expressing the following sets of strings over $\Sigma = \{0, 1\}$:
    (a) all strings starting with 101;
    (b) all strings containing the substring 000;
    (c) all strings that can be built as a products of some number of copies of the string 001; and
    (d) all strings in which a 0 is not preceded by a 1.