Matrices and transformations: in principle

In the problems below, we'll use *affine Cartesian coordinates* for $\mathbb{A}^{2+1} = \{\,(x, y, z) : x, y, z \in \mathbb{R}\,\}$, which contains the two-dimensional vectors $\langle x, y\rangle$ for $z = 0$ and the points (x, y) of the plane for $z = 1$.

1. Column vectors and matrices:
   (a) What is a ***column vector*** of some dimension k, and what is an $m \times n$ ***matrix***?
   (b) If A is an $m \times n$ matrix and $\vec{x}$ is an n-dimensional column vector, how do we define $A\vec{x}$, and how does this relate to our fundamental operation on vectors?
   How does this matrix thus act as a *function*? What are that function's domain and codomain?
   (c) If B is an $n \times p$ matrix, explain how this allows us to *compose* the matrices A and B into a new matrix AB. What is important to keep in mind about the order of action in such a composition?

2. Consider the 3×3 matrix $A = \begin{bmatrix} a & b & x_0 \\ c & d & y_0 \\ 0 & 0 & 1 \end{bmatrix}$.
   (a) Where does A map the *point* $(0, 0)$, i.e., $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$?
   (b) Where does A map the *vectors* $\langle 1, 0\rangle$ and $\langle 0, 1\rangle$, i.e., $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$?

3. How can we use *conjugation* to find a transformation that transforms the plane "about" some fixed point (x_0, y_0)?

. . . and in practice

4. Consider the following transformations of the plane:

$$T_{\langle x_0,y_0\rangle} = \begin{bmatrix} 1 & 0 & x_0 \\ 0 & 1 & y_0 \\ 0 & 0 & 1 \end{bmatrix} \qquad R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad S_r = \begin{bmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

   Determine where each of these maps the origin $(0, 0)$ and the two basic direction vectors $\langle 1, 0\rangle$ and $\langle 0, 1\rangle$, and use this to describe in words the transformation of the plane each of these matrices represent.

5. Find matrices for the following transformations of the plane, in affine coordinates:
   (a) Scale by 2 from the origin, then translate by $\langle 1, 1\rangle$.
   (b) Translate by $\langle 1, 1\rangle$, then scale by 2 from the origin.
   (c) Rotate by $\frac{\pi}{2}$ counterclockwise about the origin, then translate by $\langle 0, 2\rangle$.
   (d) Translate by $\langle 0, 2\rangle$, then rotate by $\frac{\pi}{2}$ counterclockwise about the origin.
   (e) Rotate by $\frac{\pi}{2}$ counterclockwise about the origin, then scale by 2 from the origin.
   (f) Scale by 2 from the origin, then rotate by $\frac{\pi}{2}$ counterclockwise about the origin.

   From the examples you've just seen, what do you guess about which of the basic matrices $T_{\langle x_0,y_0\rangle}$, R_θ, and S_r *commute* with each other, in general, and which don't? (Two matrices A and B ***commute*** if $AB = BA$.)

6. Find matrices for the following transformations of the plane, in affine coordinates, and check that the specified fixed point is, indeed, fixed.
   (a) Rotate the plane by $\frac{\pi}{4}$ *clockwise* about the point $(1, 3)$.
   (b) Scale the plane by 5 about the point $(2, 4)$.

7. Find a matrix for the following transformation of the plane, in affine coordinates:
   Scale by 2 from the origin, translate by $\langle -1, -1\rangle$, rotate by $\frac{\pi}{2}$ counterclockwise, then translate by $\langle 4, 0\rangle$.