Probability spaces, independence, and conditional probability: in principle

1. What are the two ingredients of a finite probability space?

2. Joint probability distributions: suppose that \((X, \mathbb{P}_X)\) and \((Y, \mathbb{P}_Y)\) are finite probability spaces. How do we define a probability distribution on the product \(X \times Y\)? How do we interpret the meaning of this new probability space? Illustrate with an example.

3. Subspaces: suppose that \(B\) is an event in some probability space \((\Omega, \mathbb{P})\), and that \(\mathbb{P}(B) > 0\).
   (a) How do the events of \(\Omega\) give us a set of events for \(B\)?
   (b) How does the probability distribution \(\mathbb{P}\) give us a probability distribution \(\mathbb{P}_B\) on \(B\)?
      What must we be careful about when doing this, and why is it so important that \(B\) had \(\mathbb{P}(B) > 0\)?

4. Suppose that \(A\) and \(B\) are events in some probability space \((\Omega, \mathbb{P})\).
   (a) What does it mean for \(A\) and \(B\) to be mutually exclusive? Illustrate with an example.
   (b) What does it mean for \(A\) and \(B\) to be independent? Illustrate with an example.

5. Suppose that \(A\) and \(B\) are events in some probability space, and that \(\mathbb{P}(B) > 0\).
   (a) How do we define the conditional probability \(\mathbb{P}(A | B)\)?
   (b) In words, what does this tell us?
   (c) How does this relate to our notion of subspaces?
   (d) What can be determined about this conditional probability if \(A\) and \(B\) are independent?

...and in practice

6. Let \((\Omega, \mathbb{P})\) be the probability space for two fair six-sided dice being rolled.
   (a) How can this probability space be considered as a product?
   (b) Compute the probabilities of each of the following, and determine which pairs of the events are independent:
      (i) The first roll is even.  (ii) The first roll is a 1, 2, or 3.
      (iii) The second roll is a multiple of 3.  (iv) The second roll is a 1, 2, or 3.
   (c) Let \(B\) be the event that the sum of the dice is between 3 and 6, inclusive.
      (i) Find the probability distribution for the the subspace given by \(B\); what adjective describes this distribution, and why?
      (ii) Determine \(\mathbb{P}(A_k | B)\) for each \(k = 1, 2, 3, 4, 5, 6\), where \(A_k\) is the event that the first roll is \(k\).
      Are any of these event \(A_k\) mutually exclusive of \(B\)? Are any independent of \(B\)?
      (iii) Determine \(\mathbb{P}(A | B)\) and \(\mathbb{P}(B | A)\), if \(A\) is the event that both rolls are equal.

7. Let \((\Omega, \mathbb{P})\) be the probability space for ten flips of a fair coin, and let \(B\) be the event that exactly 5 of the flips are \(H\).
   (a) Compute \(P(A | B)\) and \(P(B | A)\), if \(A\) is the event that the first five flips are all \(H\).
   (b) Consider the subspace of \((\Omega, \mathbb{P})\) given by \(B\),
      Compute the probabilities of the three events below in this subspace:
      (i) The first flip is \(T\).  (ii) The second flip is \(T\).  (iii) The first two flips are both \(T\).
      Are the events (i) and (ii) independent events in this subspace? Discuss.
   (c) Again consider the subspace of \((\Omega, \mathbb{P})\) given by \(B\).
      Compute the probabilities of the two events below and that of their intersection.
      (i) The first three flips are \(H\).  (ii) The last three flips are \(H\).
      What describes the relationship between these two events in this subspace?